MACHINE TOOL CHATTER SUPPRESSION USING SPINDLE SPEED VARIATION

BY

NISHANTH LINGALA

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Aerospace Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Adviser:

Professor N. Sri Namachchivaya

ABSTRACT

First part of the thesis deals with the dynamics and stability of nonlinear delay gyroscopic systems with periodically varying delay. The aim is to demonstrate that greater depths of cut may be achieved in a boring process (2 DOF) when the speed of the spindle is modulated sinusoidally instead of being kept constant. Since the variation of spindle speed is small and independent of the tool motion, by expanding the delay terms about a finite mean delay and augmenting the system, the time dependent delay system can be written as a state dependent delay system. The augmented system of equations is autonomous and has two pairs of pure imaginary eigenvalues without resonance. The center manifold and normal form methods are then used to obtain an approximate and simpler four dimensional system. Analysis of this simpler system shows that periodic variations in the delay lead to larger stability boundaries.

In [1], the authors present asymptotic expansion for the top Lyapunov exponent of a *scalar* linear delay differential equation driven by a two state markov process. We extend their result to a *vector* case. In the case when driving noise is small, we construct an asymptotic expansion for the top Lyapunov exponent, which determines the almost-sure stability of the system. In [2] authors present a technique to suppress chatter, where spindle speed is varied as piecewise constant uniform noise. Using the results of *vector* case, we attempt to see whether stabilization can be achieved by varying spindle speed as a two state markov chain. We find that the noise considered has destabilizing effect.

ACKNOWLEDGMENTS

With gratitude, I thank Professor N. Sri Namachchivaya for his guidance and support. I also thank NSF and AE department for the financial support.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

This study is concerned with suppression of chatter in machining processes. Chatter is the self-excited relative vibration between machine tool and the work piece. This results in poor surface finish, reduced dimensional accuracy and wear and tear of the machine tool.

Chatter in machining is usually attributed to Regenerative effect. The surface generated by the tool on one pass becomes the upper surface of the chip on the subsequent pass. The forces acting on the cutting tool is function of width of chip being cut. Therefore, the forces depend not only on the current state, but also on the state one revolution earlier. Important parameter governing the onset of chatter is the width of the chip being cut from the work piece (called 'width of cut' parameter). When width of cut is increased beyond a critical value, the relative vibration between machine tool and work piece increases to prohibitive values. This is the standard Hopf bifurcation scenario.

Much study has been done investigating the Hopf bifurcations associated with constant speed machining. Here we list a few: Fofana [3] examined the Hopf bifurcation of the cutting tool chatter using the centre-manifold approach, with only cubic nonlinearities. Nayfeh et al. [4] examined nonlinear cutting tool dynamics for the Hanna-Tobias model with both quadratic and cubic nonlinearities using multiple-time scales. Liang [5] considered the same model using both the Lyapunov-Schmidt and centre-manifold methods. Kalmar-Nagy et al. [6], obtained results by center-manifold method which agreed with experiments of Shi and Tobias [7]. A recent book edited by Moon [8] explores both modelling and nonlinear dynamics phenomena in material removal processes such as turning, milling, grinding and rolling.

To avoid the onset of chatter, the width of cut should be kept low. This means very low rate of material removal. Various methods have been developed to suppress chatter without reducing material-removal rates. One such method is spindle speed variation (SSV), introduced by Stoeferle and Grab [9]. Here the spindle speed is varied continuously to suppress chatter. It is attracting increasing attention because of its simplicity and effectiveness in chatter suppression (for details, see, Lin et al. [10], Tsao et al. [11], Jayaram et al. [12]). Typically the spindle speed is continuously varied through the superposition of a sinusoidally varying speed (Sinusoidal SSV or SSSV) upon the original spindle speed, the superposed sinusoidal speed being much smaller than the original speed. The amplitude and frequency of the superposed signal, to have effective suppression, can be obtained only through experimentation or simulation. It has been found in experiments conducted by Inamura and Sata [13], Hosi and Sato [14], Sexton et al. [15], and Sexton and Stone [16] that when the amplitude and the frequency of SSV are within some specific range, the chatter can be greatly suppressed or eliminated.

In a recent paper, Sri Namachchivaya and Van Roessel[17], have clarified the mechanism of chatter suppression by SSSV employing the Hanna-Tobias 1 DOF model. The suppression was shown by center-manifold and normal form methods and the effect of the amplitude and frequency of SSV was also evaluated. Such analytical results serve as an effective guide for rapidly locating stability boundaries, predicting post-critical behaviour and suppressing chatter. Having explicit formulae are very useful in the development and design of real-time control.

Motivated by the results from the above work, we study the suppression mechanism in a boring process (2 DOF). Boring process is a machining process which is usually used to enlarge and finish an existing hole that has been drilled, forged or punched. Unlike chip removal in which tool motion is predominantly one dimensional, in the boring process as the tool removes material from hole, the cutting force acting along one directional axis causes tool motion simultaneously in another direction. The equations describing the boring process would be gyroscopic with time delay terms to incorporate the regenerative effect. We apply the same techniques as in [17] and show that SSSV can be used to suppress chatter thus allowing increased width of cut. We note that such an attempt is made before by Vedula [18], but we present a more transparent derivation of the stabilization coefficient.

The effectiveness of SSSV technique depends on the amplitude and frequency of the superposed sinusoidal signal. SSSV has not been widely implemented in industry due to limitations on the range of parameters allowed by the system and the ability of the system to track the input sinusoidal signal (see [2]). Proper selection of SSSV parameters requires knowledge of dynamic characteristics of the system. The above mentioned work [17] gives a good direction in this regard. Also, methods have been proposed which suppress chatter by varying spindle speed in a step fashion instead of continuous signals. Yilmaz et al. [2] proposes a chatter suppression technique called Multi-level Random Spindle Speed Variation (MRSSV). In this technique, spindle speed is varied as piece-wise constant with jumps spaced uniformly in time. The amplitude of the signal is generated by a uniform distributed noise. The duration of time step is determined by the system tracking properties and the amplitude, usually not exceeding 40 percent of the nominal speed, is determined by spindle system characteristics. Using finite difference scheme for numerical simulation and also by experiments the authors show that stability boundary can be enlarged.

There exists no analytical results that demonstrate chatter suppression using random spindle speed variation. Recently, Namachchivaya and Wishtutz [1] have obtained asymptotic expansion for the top Lyapunov exponent of a scalar noisy process governed by delay differential equation with noise being a two state markov chain. Motivated by this result, we extend the result to vector case. We construct an asymptotic expansion for the top Lyapunov exponent, which determines the almost-sure stability of the system. Using this result, we attempt to see whether stabilization can be achieved by varying spindle speed as a two state markov chain. We find that such a variation would have destabilizing effect.

The work in this thesis is heavily based on [1], and [17].

In the rest of this chapter, we present the required mathematical framework.

1.1 Delay Differential Equations

The regenerative effect entails that the equations governing the dynamics are delay differential equations (DDE). That is evolution of the system depends not only on the present state but also the past. We assume that the cutting force is proportional to the width of chip being cut. So, if $x(t)$ represents tool position at time t and $x(t-r)$ the position one revolution ago, then the cutting force is proportional to $x(t) - x(t-r)$. When the past dependence is in terms of only the state variable but not its derivatives, then the equation is said to be of Retarded type. Here we illustrate the required framework for the case of a simple scalar retarded DDE. Both the problems dealt with in this thesis are vector valued DDE, but we hope that this presentation would give the essence of the framework. The material presented in this section can be found in the book [19].

Consider the following simple scalar RDDE, where a, b and $r > 0$ are constants and f a continuous function on $\mathbb{R}$:

$$\dot{x}(t) = ax(t) + bx(t-r) + f(t) \tag{1.1}$$

Given a continuous function $\phi(t)$ on $[-r, 0]$, then the above equation has a unique solution such that $x(t) = \phi(t)$ on $[-r, 0]$; the derivative at $t = 0$ being the right-hand derivative.

Note that for the evolution of system at time t, the only information relevant is from $[t-r, t]$. Therefore, one can think of solutions from a different viewpoint. Consider a window of length r. Initially the right corner of this window is placed at $t = 0$. So one can see the initial function $\phi(t)$ in the window. As the system evolves to t, the right corner of the window is moved to t. So one can view the solution $x(t)$ in the interval $[t-r, t]$. If one is travelling with the window, all that can be seen is one continuous function changing smoothly to a different continuous function. The following formalizes this idea.

A continuous solution x of (1.1) is an element of, $C([-r, \infty), \mathbb{R})$, the space of continuous functions mapping $[-r, \infty]$ to $\mathbb{R}$. The segment of the solution in the window is an element in Banach space $\mathcal{C}$ of continuous functions from the interval $[-r, 0]$ to $\mathbb{R}$, endowed with sup norm i.e. $\mathcal{C} \stackrel{\text{def}}{=} C([-r, 0], \mathbb{R})$. If the solution $x \in C([-r, \infty), \mathbb{R})$, then for any $t \in [0, \infty)$, we let $x_t \in \mathcal{C}$ be defined by

$$x_t(\theta) \stackrel{\text{def}}{=} x(t + \theta), \quad -r \leq \theta \leq 0$$

$x_t(\theta)$ is the $[t-r, t]$ segment of the solution.

Now consider (1.1) with $f \equiv 0$. It can be written as

$$\dot{x}(t) = ax_t(0) + bx_t(-r) \tag{1.2}$$

Let $Lx_t \overset{\text{def}}{=} ax_t(0) + bx_t(-r)$. Then $L : \mathcal{C} \to \mathbb{R}$ is a linear functional on $\mathcal{C}$. We could have considered the general case

$$\dot{x}(t) = Lx_t \tag{1.3}$$

where $L : \mathcal{C} \to \mathbb{R}^n$ is a linear functional on $\mathcal{C}$, with $\mathcal{C} \overset{\text{def}}{=} C([-r, 0], \mathbb{R}^n)$. By Reisz theorem, there exists a $n \times n$ matrix $\eta(\theta)$, $-r \leq \theta \leq 0$, whose elements are of bounded variation, normalized so that η is continuous from left on $(-r, 0)$ and $\eta(0) = 0$, such that

$$L\varphi = \int_{-r}^{0} d[\eta(\theta)]\varphi(\theta) \qquad \varphi \in \mathcal{C}$$

For example, for (1.2), this $\eta(\theta)$ is given as

$$\eta(\theta) = \begin{cases} 0 & \theta = 0 \\ a & -r < \theta < 0 \\ a - b & \theta = -r \end{cases}$$

We said before that if one travels with the window, one can see a continuous function changing smoothly to a different continuous function. This transformation is done by what is called a semigroup. Also, the transformation in infinitesimal time can be described by what is called infinitesimal generator. The following formalizes these ideas.

A strongly continuous semigroup of linear operators is a one parameter family $T(t) : \mathcal{C} \to \mathcal{C}, t \geq 0$ of bounded linear operators that satisfy the following properties:

- $T(0) = I$
- $T(s + t) = T(s)T(t)$
- $\lim_{t \downarrow 0} ||T(t)\varphi - \varphi|| = 0 \qquad \varphi \in \mathcal{C}$

We have only listed the properties of a semigroup. We have not yet obtained one for (1.2). If $x_t(., \phi)$ is the unique solution to (1.2) with initial condition ϕ, then $x_t(., \phi) = T(t)\phi$.

Infinitesimal generator $A : \mathcal{D}(A) \to \mathcal{C}$ of a strongly continuous semi-group

$T(t)$ is defined by

$$A\varphi = \lim_{t \downarrow 0} \frac{T(t)\varphi - \varphi}{t} \tag{1.4}$$

The domain $\mathcal{D}(A)$ is not yet specified. It can be shown that for every $\varphi \in \mathcal{D}(A)$, $t \to T(t)\varphi$ satisfies the differential equation

$$\frac{d}{dt}T(t)\varphi = AT(t)\varphi = T(t)A\varphi$$

Now we show what the solution operator $T(t)$ and the infinitesimal generator A are for (1.2).

$$T(t)\phi(\theta) = \begin{cases} \phi(t+\theta) & t+\theta \leq 0 \\ \phi(0) + \int_0^{t+\theta} L(T(s)\phi)ds & t+\theta > 0 \end{cases}$$

Using the above formula, one can see that, in RHS of 1.4, if $\theta < 0$, the limit is $\frac{d\phi}{d\theta}$, and if $\theta = 0$, then the limit is $L\phi$. For $\dot{x}(t)$ to be well-defined, we need $L\phi = \frac{d\phi}{d\theta}|_{\theta=0}$. Therefore, the infinitesimal generator is given by

$$\begin{aligned} A\phi &= \frac{d\phi}{d\theta} && (1.5) \\ \mathcal{D}(A) &= \left\{ \phi \in \mathcal{C} : \frac{d\phi}{d\theta} \in \mathcal{C} \text{ and } \frac{d\phi}{d\theta}|_{\theta=0} = L\phi \right\} && (1.6) \end{aligned}$$

Next we deal with spectrum of A and the generalized eigen-spaces. Spectrum determines the long-term behaviour of solution. For the problems dealt in this thesis, there exists two eigenvalues on the imaginary axis indicating periodic solution. If all the other eigenvalues lie in left half of complex plane, then trajectories quickly reach the eigenspace corresponding to the imaginary eigenvalues. Note that the original DDE is infinite dimensional. But when the dynamics is projected onto this eigenspace, system governing the dynamics would be $\mathbb{R}^2$ valued ODE.

For any operator $A : \mathcal{D}(A) \to \mathcal{C}$, the resolvent set $\rho(A)$ is the set of values λ in $\mathbb{C}$ for which the operator $\lambda I - A$ has a bounded inverse with domain dense in $\mathcal{C}$. The spectrum $\sigma(A)$ is then defined as $\sigma(A) = \mathbb{C} \setminus \rho(A)$. Generalized eigenspace of λ, $\mathcal{M}_\lambda(A)$, is the smallest subspace of $\mathcal{C}$ containing all elements of $\mathcal{C}$ belonging to ker $\left((\lambda I - A)^k\right)$. Dimension of $\mathcal{M}_\lambda(A)$ is minimum k such that ker $\left((\lambda I - A)^k\right)$=ker $\left((\lambda I - A)^{k+1}\right)$.

The following properties can be proved:

- A has only point(discrete) spectrum. $\lambda \in \sigma(A)$ iff λ satisfies the characteristic equation $\lambda I - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta) = 0$.

- $\mathcal{M}_\lambda(A)$ is finite dimensional for any $\lambda \in \sigma(A)$.

- $\mathcal{C} = \ker\left((\lambda I - A)^k\right) \oplus R\left((\lambda I - A)^k\right)$, where $k = \dim \mathcal{M}_\lambda(A)$

- $\mathcal{M}_\lambda(A)$ satisfies $A\mathcal{M}_\lambda(A) \subset \mathcal{M}_\lambda(A)$. Let $\dim \mathcal{M}_\lambda(A) = d$, and let $\phi_1^\lambda, \ldots, \phi_d^\lambda$ be a basis for $\mathcal{M}_\lambda(A)$. Let $\Phi_\lambda = \{\phi_1^\lambda, \ldots, \phi_d^\lambda\}$. Then there is a $d \times d$ constant matrix B_λ such that $A\Phi_\lambda = \Phi_\lambda B_\lambda$. Further, only eigenvalue of B_λ is λ.

Given any $\phi \in \mathcal{C}$, we can write it as $\phi^P + \phi^Q$ where $\phi^P \in \ker\left((\lambda I - A)^k\right)$ and $\phi^Q \in R\left((\lambda I - A)^k\right)$. We will make use of the fact that $R\left((\lambda I - A)^k\right) = \ker\left((\lambda I - A^*)^k\right)^\perp$, where A^* is adjoint operator. So we need to consider the adjoint operator.

If $T(t)$ is the solution semigroup corresponding to (1.3), then $T(t,s), t \geq s$ defined by $T(t,s)x_s(\theta) = x_t(\theta)$ is called forward evolutionary system corresponding to (1.3) on $\mathcal{C}$.

Let $\mathcal{C}^*$ denote the Banach space of row-valued functions $\psi : (-\infty, 0] \to \mathbb{R}^{n*}$ that are constant on $(-\infty, -r]$, of bounded variation on $[-r, 0]$, continuous from left on $(-r, 0)$ and vanishing at zero with norm $\mathrm{Var}_{[-r,0]}\,\psi$. With the pairing

$$\langle \psi, \phi \rangle = \int_{-r}^{0} d\psi(\theta)\phi(\theta), \qquad \psi \in \mathcal{C}^*, \quad \phi \in \mathcal{C}$$

the space $\mathcal{C}^*$ is the dual space of $\mathcal{C}$.

Let $\phi \in \mathcal{C}$ and $g \in \mathcal{C}^*$. Then a two parameter family $V(s,t), s \leq t$ defined by $\langle \phi, V(s,t)g \rangle = \langle T(t,s)\phi, g \rangle$ induces a semigroup $T^*(s)$ on $\mathcal{C}^*$. This is called the adjoint semigroup. It can be shown that $V(s,t)$ and $T^*(s)$ corresponds to the Volterra equation

$$y(s) + \int_{0}^{s} y(\tau)\eta(s-\tau)d\tau = g(s), \quad s \leq 0$$

with $g \in \mathcal{C}^*$, η corresponding to (1.3). $V(s,t)$ is backward evolution system of the above Volterra equation. It can be shown that $T^*(s)$ is not a strongly continuous operator i.e. the third property listed for semigroups is not satisfied. Hence, one cannot define infinitesimal generator for this semigroup. Instead, the adjoint A^* of A is defined as follows: $f \in \mathcal{D}(A^*)$ iff $\exists g \in \mathcal{C}^*$

such that $\langle f, A\phi \rangle = \langle g, \phi \rangle$ for all $\phi \in \mathcal{D}(A)$ and in that case $A^* f = g$. With this definition, it can be shown that the adjoint operator $A^* : \mathcal{D}(A^*) \to \mathcal{C}^*$ is given by

$$\begin{aligned}
\mathcal{D}(A^*) &= \left\{ f \in \mathcal{C}^* : \frac{df}{d\theta} \in \mathcal{C}^* \right\} \\
A^* f(\theta) &= f(0-)\eta(\theta) - \frac{df}{d\theta}(\theta), \qquad -r \leq \theta \leq 0
\end{aligned}$$

A closely related equation to (1.3) is what is called transposed equation. The transpose of the system (1.3) is the following system:

$$\dot{y}(s) = \int_{-r}^{0} y(s-\theta) d\eta(\theta), \quad s \leq 0$$

with initial condition $y(s)$ specified in $[0, r]$.

$$\mathcal{C}' \stackrel{\text{def}}{=} C([0, r], \mathbb{R}^{n*})$$

For each $s \in [0, \infty)$ let $y^s(\xi) \stackrel{\text{def}}{=} y(-s+\xi), 0 \leq \xi \leq r$. One can associate a strongly continuous semigroup $T^T(t)$ with the above equation, for which the infinitesimal generator A^T is given by

$$\begin{aligned}
\mathcal{D}(A^T) &= \left\{ \psi \in \mathcal{C}' : \frac{d\psi}{d\xi} \in \mathcal{C}', \frac{d\psi}{d\xi}(0) = -\int_{-r}^{0} \psi(-\theta) d\eta(\theta) \right\} \\
A^T \psi &= -\frac{d\psi}{d\xi}
\end{aligned}$$

The relation between the adjoint semigroup and transpose semigroup is the following. Consider $F^T : \mathcal{C}' \to \mathcal{C}^*$, given by

$$(F^T\psi)(s) = \psi(0) - \int_{s}^{0} \int_{-r}^{\alpha} \psi(\alpha - \theta) d\eta(\theta) d\alpha$$

Then, $F^T T^T(s)\psi = T^*(s) F^T \psi$ It can also be shown that A^* and A have the same specturm and that $A^* F^T \psi = F^T A^T \psi \quad \forall \psi \in \mathcal{D}(A^T)$ i.e. if ψ is a generalized eigenfunction of A^T, then $F^T\psi$ is a generalized eigenfunction of A^*.

Note that we were interested in the adjoint operator because we wanted to use the fact that $R\left((\lambda I - A)^k\right) = \ker\left((\lambda I - A^*)^k\right)^{\perp}$. But $\ker\left((\lambda I - A^*)^k\right)$ is the space spanned by eigenfunction of A^*, or by the above relation, the space spanned by functions of the form $F^T\psi$ where ψ are eigenfunctions of A^T.

Suppose we introduce the following pairing between $\mathcal{C}$ and $\mathcal{C}'$: For $\psi \in \mathcal{C}'$ and $\phi \in \mathcal{C}$,

$$\begin{aligned}(\psi,\phi) \overset{\text{def}}{=} \left\langle F^T\psi,\phi\right\rangle &= \int_{-r}^{0} d[F^T\psi(\theta)]\phi(\theta)\\ &= \psi(0)\phi(0) - \int_{-r}^{0}\int_{0}^{\theta}\psi(\theta-\tau)d\eta(\tau)\phi(\theta)d\theta\end{aligned}$$

Then it can be shown that with respect to this bilinear form, A^T satisfies $(\psi, A\phi) = (A^T\psi, \phi)$. Therefore A^T acts like an adjoint with this pairing. Therefore we have the following result:

For $\lambda \in \sigma(A)$, let $p = \dim \mathcal{M}_\lambda(A)$. Let Ψ_λ be the matrix whose columns form a basis for $\mathcal{M}_\lambda(A^T)$ and Φ_λ be the matrix whose rows form a basis for $\mathcal{M}_\lambda(A)$. Let $(\Psi_\lambda, \Phi_\lambda) = (\psi_i, \phi_j)$ for $i, j = 1 \ldots p$. $(\Psi_\lambda, \Phi_\lambda)$ is nonsingular and may be taken as identity. Let

$$\begin{aligned}P_\lambda &= \mathcal{M}_\lambda(A) = \{\phi \in \mathcal{C} : \phi = \Phi_\lambda b \text{ for some p-vector}\, b\}\\ Q_\lambda &= \{\phi \in \mathcal{C} : (\Psi_\lambda, \phi) = 0\}\end{aligned}$$

Then, any $\phi \in \mathcal{C}$ can be split as $\phi = \phi^{P_\lambda} + \phi^{Q_\lambda}$, where $\phi^{P_\lambda} \in P_\lambda$ and $\phi^{Q_\lambda} \in Q_\lambda$. $\phi^{P_\lambda} = \Phi_\lambda(\Psi_\lambda, \phi)$ and $\phi^{Q_\lambda} = \phi - \phi^{P_\lambda}$. Further, there exists positive constant κ, γ such that

$$||T(t)\phi^{Q_\lambda}|| \le \kappa e^{-\gamma t}||\phi^{Q_\lambda}||$$

In the hopf bifurcation scenario that we consider, this inequality says that the dynamics quickly falls onto the generalized eigenspace of the imaginary eigenvalues.

Now, consider the perturbed nonlinear problem. What is presented below is found in [20].

$$\dot{x}(t) = Lx_t + F(x_t), \quad x_0 = \phi \in \mathcal{C} \tag{1.7}$$

Making use of the semigroup from unperturbed problem, the above equation can be written in integrated form as

$$x_t = T(t)\phi + \int_0^t T(t-s)X_0 F(x_s)ds \tag{1.8}$$

where $X_0 = X_0(\theta)$ is given by

$$X_0(\theta) = \begin{cases} I_{n\times n} & \theta = 0 \\ 0 & -r \leq \theta_0 \end{cases}$$

Note that $X_0 \notin \mathcal{C}$. If (1.8) is differentiated with respect to t, we obtain the formal expression

$$\frac{d}{dt}x_t = Ax_t + X_0 F(x_t)$$

Note that though $x_t \in C^1$, $x_t \notin \mathcal{D}(A)$ because x_t does not satisfy $\frac{dx_t}{d\theta}|_{\theta=0} = Lx_t$. Hence Ax_t doesn't make any sense in the above equation. Also note that $X_0 \notin C$. This situation is remedied in the following way. Note that any ϕ that doesn't satisfy $\frac{d\phi}{d\theta}|_{\theta=0} = L\phi$, but $\phi \in C^1$ can be decomposed as $\phi = \phi^1 + \phi^2$ where ϕ^1 satisfies $\frac{d\phi^1}{d\theta}|_{\theta=0} = L\phi^1$ and ϕ^2 is a constant function. Also, note that the constant function $\left[\int_{-r}^0 d\eta(\theta)\right]^{-1}$ satisfies $A\left[\int_{-r}^0 d\eta(\theta)\right]^{-1} = X_0$. So, to attain the above decomposition, ϕ^2 can be defined by

$$\left.\frac{d\phi^1}{d\theta}\right|_{\theta=0} = L\phi^1 = L(\phi - \phi^2) = L\phi - \left[\int_{-r}^0 d\eta(\theta)\right]\phi^2$$

Then, we have

$$A\phi = A\phi^1 + A\phi^2 = \frac{d\phi}{d\theta} + X_0\left(L\phi - \left.\frac{d\phi}{d\theta}\right|_{\theta=0}\right)$$

Let

$$\begin{aligned} \mathcal{BC} &\overset{\text{def}}{=} \mathcal{C} \oplus \langle X_0 \rangle \\ &= \{\phi : [-r,0] \to \mathbb{R}^n, \phi\,\text{continuous on}[-r,0)\text{with jump at }0\} \end{aligned}$$

$$\mathcal{C}^1 = \left\{\phi : \phi \in \mathcal{C}, \frac{d\phi}{d\theta} \in \mathcal{C}\right\}$$

Any element in $\mathcal{BC}$ is of the form $\phi + X_0\alpha$ for some $\phi \in \mathcal{C}$. With the norm

$||\phi + X_0\alpha||_{\mathcal{BC}} = ||\phi||_{\mathcal{C}} + ||\alpha||_{\mathbb{R}^n}$, $\mathcal{BC}$ is a Banach space. Now, define a new map $\hat{A} : \mathcal{C}^1 \to \mathcal{BC}$, defined by $\hat{A}\phi = \frac{d\phi}{d\theta} + X_0(L\phi - \frac{d\phi}{d\theta}|_{\theta=0})$. Then solution of (1.7) satisfies

$$\frac{dx_t}{dt} = \hat{A}x_t + X_0F(x_t)$$

The previous bilinear form can be extended to $\mathcal{C}' \times \mathcal{BC}$ by setting $(\Psi, X_0) = \Psi(0)$. It can be shown that A and $\hat{A}$ have the same spectrum. Analogous to the previous projection (on to P_λ), define the projection operator $\pi : \mathcal{BC} \to P$ as

$$\pi(\phi + X_0\alpha) = \Phi[(\Psi, \phi) + \Psi(0)\alpha]$$

Then $\mathcal{BC} = P \oplus \operatorname{Ker}\pi$. It can be shown that π commutes with $\hat{A}$ in $\mathcal{C}^1$. Hence, for the hopf bifurcation scenario we are interested in, we can write $x_t = \Phi z(t) + y_t$ where $z(t) \in \mathbb{R}^2$ and $y_t \in \operatorname{Ker}\pi \cap \mathcal{D}(\hat{A}) = Q \cap \mathcal{C}^1 \stackrel{\text{def}}{=} Q^1$ and Φ being the basis for the eigenspace corresponding to the eigenvalues $\pm i\omega$. Consider the equation

$$\dot{x}(t) = Lx_t + F(x_t) \tag{1.9}$$

Suppose with $F \equiv 0$, the system $\dot{x}(t) = Lx_t$ has a pair of eigenvalues on the imaginary axis and rest of the eigenvalues have negative real parts. Then, the system $\dot{x}(t) = Lx_t + F(x_t)$ is equivalent to the following system obtained by projecting onto P.

$$\dot{z} = Bz + \Psi(0)F(\Phi z + y_t) \tag{1.10}$$

$$\frac{d}{dt}y_t = \hat{A}_{Q^1}y + (I - \pi)X_0F(\Phi z + y_t) \tag{1.11}$$

First equation in the above system describes the dynamics projected onto the eigenspace of the imaginary eigenvalues.

1.2 Random Dynamical Systems (RDS)

In chapter 3 of this thesis, we study the dynamics of machine tool when spindle speed is varied randomly. The machine tool constitutes a system and it is perturbed by a noise.

RDS consists of two ingredients: a model of the noise, and a model of the system perturbed by noise. Model of the noise is given by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a flow of transformations $\theta_t : \Omega \to \Omega$. Ω consists of all the possible outcomes of the noise process. $\mathcal{F}$ is the set of all events possible with the noise and $\mathbb{P}$ is a measure which assigns probability to the events in $\mathcal{F}$. The flow θ_t specifies how the outcomes of noise change with time. It satisfies $\theta_0 = id$, $\theta_{t+s} = \theta_t \circ \theta_s$ and it doesn't alter how the probabilities are assigned i.e. $\theta_t \mathbb{P} = \mathbb{P}$ i.e. it is a measure preserving flow.

Dynamics in the phase space of a system perturbed by noise is given by a smooth mapping $\varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $(t, \omega, x) \mapsto \varphi(t, \omega)x$. In absence of noise, the dynamics should be smooth and so $(t, x) \mapsto \varphi(t, \omega)x$ should be continuous in (t, x) and smooth in x. Also φ satisfies $\varphi(0, \omega) = id_{\mathbb{R}^d}$ and $\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega)$. The last propery is called cocylce property and hence φ is called a *cocyle* over θ.

The flow corresponding to the dynamics on $\Omega \times \mathbb{R}^d$ is then given by $\Theta_t(\omega, x) \stackrel{\text{def}}{=} (\theta_t \omega,\ \varphi(t, \omega)x)$. This Θ_t is called the *skew product flow* corresponding to φ.

Let μ be a probability measure on $\Omega \times \mathbb{R}^d$. μ changes with time i.e. for each time t, $\int_A \mu$ gives the probability that the system is in $A \subset \Omega \times \mathbb{R}^d$. μ changes with time according to the flow Θ_t.

μ is called *invariant* for φ if $\Theta_t \mu = \mu$ for all $t \in \mathbb{R}$ and the marginal of μ on Ω is $\mathbb{P}$. The last part of the above statement is due to the fact that the noise process is *given* to us and we have no control over it. Note that RDS has a $\theta-$invariant $\mathbb{P}$, but does not in general come equipped with an invariant measure. Also, an invariant measure need not be a product measure.

However, an invariant measure can be uniquely factorized, $\mu(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega)$, where $(\omega, B) \mapsto \mu_\omega(B)$ is a probability measure on $\mathbb{R}^d$ for each fixed ω. We have for all measurable $A \subset \Omega \times \mathbb{R}^d$,

$$\mu(A) = \int_\Omega \int_{\mathbb{R}^d} \mathbf{1}_A(\omega, x)\mu_\omega(dx)\mathbb{P}(d\omega)$$

For an RDS with $t \mapsto \phi(t, \omega)x$ a Markov process with transition probability $P(t, x, B)$ and generator L, a measure ρ on $\mathbb{R}^d$ is called *stationary* if it satisfies for all t,

$$\rho(\cdot) = \int_{\mathbb{R}^d} P(t, x, \cdot)\rho(dx),$$

equivalently, if it solves the Fokker-Planck equation $L^* \rho = 0$.

There is a one-one correspondence between stationary ρ's and those invariant μ_ω's which are measurable with respect to $\sigma-$algebra of the past noise. However, there are in general more invariant measures than those coming from stationary measures.

Consider the Jacobian of $\varphi(t,\omega)$ at x,

$$D\varphi(t,\omega,x) \stackrel{\text{def}}{=} \left(\frac{\partial\left(\varphi(t,\omega,x)\right)_i}{\partial x_j}\right)$$

D is a cocylce over Θ_t. D satisfies

$$D\varphi(t+s,\omega,x) = D\varphi(t,\Theta_s(\omega,x))D\varphi(s,\omega,x)$$

Consider any point $x \in \mathbb{R}^d$. Consider any point v in the tangent space $T_x\mathbb{R}^d$ at x (note that we are dealing with $\mathbb{R}^d$, and so for any point $x \in \mathbb{R}^d$, the tangent space is again $\mathbb{R}^d$, except that the origin of this $\mathbb{R}^d$ is centered at x) i.e. initial separation between the two points is v. Denote by

$$\lambda(\omega,x,v) \stackrel{\text{def}}{=} \lim_{t\to\infty}\frac{1}{t}\log||D\varphi(t,\omega,x)v||$$

the Lyapunov exponent of v at x. This Lyapunov exponent gives the rate of separation of trajectories starting from x with initial separation v.

According to Multiplicative Ergodic Theorem, there exists a list of fixed numbers $\lambda_1 > \ldots > \lambda_p$ with multiplicities d_i, $\sum_{i=1}^{p} d_i = d$, such that the tangent space $T_x\mathbb{R}^d \cong \mathbb{R}^d$ at any x splits into a direct sum of measurable subspaces i.e.

$$T_x\mathbb{R}^d \cong \mathbb{R}^d = E_1(\omega,x) \oplus \ldots E_p(\omega,x)$$

which are invariant i.e. $D\varphi(t,\omega,x)E_i(\omega,x) = E_i(\Theta_t(\omega,x))$, with $\dim E_i(\omega,x) = d_i$. This splitting is characterized as follows: if $v \in E_i(\omega,x)$, then the separation between trajectories that start from x with initial separation v and initial noise ω increases at the rate λ_i. Also, if $v = \oplus_{i=1}^{p} v_i$, with $v_i \in E_i(\omega,x)$, then separation grows at the rate $\lambda_{i_0}(\omega,x,v)$ where $i_0 = \min\{i : v_i \neq 0\}$. The Lyapunov exponents and the subspaces described depends on φ as well as μ.

If the top Lyapunov exponent $\lambda_1 < 0$ then the flow φ is stable under μ, i.e. distance between trajectories that are intially closely separated goes to zero exponentially fast with rate λ_1.

For the problem that we are dealing with in chapter 3, we are interested in the stability and hence in the top Lyapunov exponent. Since exact invariant measure is difficult to solve for, we construct asymptotic expansion for invariant measure. We use the Furstenberg-Khasminskii formula (presented in 3.3)for finding top Lyapunov exponent.

1.3 Continuous Time Markov Chains

In this section we present the notion of infinitesimal generator and invariant measure of a continuous time, two state markov chain. Consider a stationary ergodic two-state Markov process $\xi(t)$ with state space $\mathbf{M} \stackrel{\text{def}}{=} \{1, 2\}$ and transition intensities $1 \xrightarrow{g_{12}} 2$, $2 \xrightarrow{g_{21}} 1$ i.e.

$$Pr\{\xi(t+h) = j|\xi(t) = i\} = g_{ij}h + o(h) \qquad i \neq j$$

Consider the matrix given by

$$G = \begin{pmatrix} -g_{12} & g_{12} \\ g_{21} & -g_{21} \end{pmatrix}$$

The eigenvalues of G are $-(g_{12} + g_{21})$ and 0, and the corresponding eigenvectors are $[g_{12}, g_{21}]^T$ and $[1, 1]^T$. Now, let $P(t)$ be the transition probability matrix defined as:

$$Pr\{\xi(t+s) = j|\xi(s) = i\} = P_{ij}(t)$$

Then $P(t)$ satisfies the differential equation $\frac{dP}{dt} = GP(t)$, with initial condition $P(0) = I$ which can be solved as $P(t) = e^{Gt}$. Suppose B is a matrix whose columns are the eigenvectors of G and Γ be a diagonal matrix with eigenvalues of G as its entries. Then $G = B\Gamma B^{-1}$, and

$$P(t) = e^{Gt} = e^{B\Gamma B^{-1}t} = Be^{\Gamma t}B^{-1}$$

which can be evaluated as

$$P(t) = \frac{1}{g_{12} + g_{21}} \begin{bmatrix} g_{12}e^{-(g_{12}+g_{21})t} + g_{21} & g_{12}(1 - e^{-(g_{12}+g_{21})t}) \\ g_{21}(1 - e^{-(g_{12}+g_{21})t}) & g_{21}e^{-(g_{12}+g_{21})t} + g_{12} \end{bmatrix}$$

Now consider the family of operators $\{U_t, t \geq 0\}$ transforming a bounded continuous function f on $\mathbf{M}$ into $U_t f$ by

$$(U_t f)(i) \stackrel{\text{def}}{=} \mathbb{E}[f(\xi(t))|\xi(0) = i] = \sum_{j=1}^{2} f(j) P_{ij}(t)$$

$U_t f$ can be evaluated to be

$$\begin{aligned}(U_t f)(1) &= \frac{1}{g_{12} + g_{21}} \left[g_{12} e^{-(g_{12}+g_{21})t}(f(1) - f(2)) + g_{21} f(1) + g_{12} f(2)\right] \\ (U_t f)(2) &= \frac{1}{g_{12} + g_{21}} \left[g_{21} e^{-(g_{12}+g_{21})t}(f(2) - f(1)) + g_{21} f(1) + g_{12} f(2)\right]\end{aligned}$$

From the above equations it can be seen that $\lim_{t\to 0} U_t = I$ and that $\frac{dU_t}{dt} = GU_t$. For these reasons, G is called the infinitesimal generator of the $\xi(t)$ process. Note that

$$\lim_{t\to\infty} U_t f(1) = \frac{1}{g_{12} + g_{21}} (g_{21} f(1) + g_{12} f(2)) = \lim_{t\to\infty} U_t f(2)$$

Let $\nu = \frac{1}{g_{12}+g_{21}}[g_{21}, g_{12}]$. Then

$$\lim_{t\to\infty} U_t f(1) = \nu[f(1), f(2)]^T = \lim_{t\to\infty} U_t f(2)$$

Hence ν is the invariant measure. It can also be checked by noting that $P^T(t)\nu = \nu$.

1.4 Structure of the Thesis

Rest of the thesis is organized in the following manner.

Aim of Chapter 2 is to show that SSSV technique can be used to suppress chatter in a boring process. We study the dynamics and stability of 2 DOF gyroscopic system with delay, when the delay has small periodic fluctuations. A brief account of previous research done in dynamics and stability of boring process is given in the introduction. Model of the boring process and the governing equations are presented in 2.1. Then the problem is formulated in functional differential equation (FDE) framework in 2.2. At the onset of chatter, the system has a pair of eigenvalues on the imaginary axis and rest of the eigenvalues have negative real part. The trajectories are quickly attracted to the center manifold. We illustrate the center manifold reduction in 2.3 and normal form in 2.4. Then we show that periodic fluctuations in delay results in larger stability boundaries. All the calculations are performed in the appendix A.

Aim of Chapter 3 is to extend the result of [1] to the case of vector linear delay differential equation. A general case would be

$$\dot{x}(t) = Ex(t) + Dx(t-r) + \varepsilon\left(Fx(t) + Hx(t-r)\right)\sigma(\xi(t)) \tag{1.12}$$

where $x(t) = \{x_1(t),\, x_2(t)\}^T$ and

$$E = \begin{bmatrix} 0 & 1 \\ -(1+\kappa) & -2\zeta \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ \kappa & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

and $\sigma(\xi)$ is a mean zero function of the noise ξ. In the absence of noise term, for a fixed r, $\exists\, \kappa_c$ such that, the $(0,0)$ solution of the above system is stable for $\kappa < \kappa_c$, unstable for $\kappa > \kappa_c$. For $\kappa = \kappa_c$ the system exhibits periodic solutions. Our aim is also to check whether chatter suppression is possible by modulating spindle speed as a two state markov process. We present the model of machining in 3.1. This conforms to a special case of the above system. We focus on the special case and formulate the problem in FDE framework in 3.2. Then we study the stochastic stability of the $(0,0)$ solution and illustrate a procedure to find asymptotic expansion of the invariant measure in 3.4. We compute the top Lyapunov exponent (to ϵ^2 order) for the case of two state markov chain in 3.5. Then we attempt to apply our result for the chatter suppression problem.

CHAPTER 2

STABILITY OF GYROSCOPIC SYSTEMS WITH PERIODICALLY VARYING DELAY

In this chapter we show that SSSV technique can be used to suppress chatter in a boring process. A brief account of some of the research done in dynamics and stability of boring process will be given. In section 2.1 a model of boring process is presented. Then SSV is incorporated in the model. Functional Differential Equation (FDE) framework in which the problem is formulated is given in 2.2. Center manifold reduction and computation of normal form is done in sections 2.3 and 2.4 respectively. After performing the stability analysis in section 2.5 we apply the result to a specific operating condition and show the increase in stability boundary. All the algebraic manipulations that need to be done in the above sections are deferred to appendix.

Various models have been proposed for the boring porcess. Tobias and Fishwick [21] proposed that, under chatter conditions, the cutting force is a function not only of the chip thickness but also of the penetration rate and cutting speed variation. Hallam and Allsopp [22] during the design of a dynamometer for measuring forces on the tool, developed a simple mechanistic boring force model so as to be able to predict the realtionship between the force on the tool and various parameters. In this simple model it was assumed that the cutting force is proportional to the cross-section of the uncut chip area. When compared with experimental data, their model showed good agreement of the relationship between the force components and the depth of cut but the agreement was not good for the relationship between force components and feed rate. This indicates that cutting force is a function not only of the chip thickness as Tobias proposed.

For boring process in which boring bar is stationary and the work piece rotates, Tlutsy [23] considered a simple practical system for the case of two orthogonal modes of vibration and developed a mathematical model in which two second-order differential equations were used to describe the boring bar motion. The slender boring bar is represented at the tool point by a 2 DOF mass-spring-damper system. Based on this model, Tlutsy and Polacek [24], developed the principles of mode coupling and regenerative effects. Mode coupling is a self-excitation mechanism that exists when the relative motion between tool and work piece is possible in two directions perpendicular to the tool axis. If motion in two directions is of same frequency but with a phase shift, then an elliptical motion of the tool tip occurs. During first half of the cycle, force from the work piece acts in a direction opposite to the tool motion and energy of the tool motion is removed. But during the later half, the force from the work piece is in the same direction as the tool motion and energy is added. But because tool is in deeper part of the work piece in the later half of the cycle, more energy is added than is taken away. This energy would sustain the motion against damping losses. Parker [25] has included the effect of penetration rate in Tlutsy's model. In this model, cutting force is assumed to have two components: one component is proportional to thickness of chip being cut, and other component is proportional to penetration velocity. In the experiments that were performed, however, instability which is characterized by vibrations solely in direction tangential to the machined surface has been observed. Experiments performed suggested that a significant increase in cutting force is observed when an increase in tangential vibration occured. Zhang and Kapoor [26] developed a system model which incorporated this effect. They assumed that cutting force has an additional component proportional to the magnitude of tangential vibration. Pratt [27] studied the linear stability and nonlinear dynamics, including both quadratic and cubic nonlinearities, of a two DOF model representing the motion of a boring bar using the method of multiple time scales.

Most previous research on stability analysis for the boring process has focused on the case where the boring bar is stationary while the workpiece rotates. Thus, the direction of the forces on the machine tool are fixed with respect to an inertial frame. However, for some boring processes such as a line boring process, the tool is rotating and the workpiece is stationary and the directions of the radial and tangential forces are not fixed with respect to an inertial frame and rotate along with the boring bar.

The analysis for the rotating tool problem may be performed in either the inertial or rotating frames. In the former case, the coefficients in the differential equation are periodic and a Fourier series expansion may be used to obtain an approximate result. In the latter case, it is possible to obtain an exact solution since the coefficients in the differential equation are time-invariant. However, care must be taken to account for the dynamic, spindle-speed dependent coordinate coupling.

Chen and Wang [28] addressed the stability analysis of rotating tools using discretized version of a distributed parameter model. They formulated the equations of motion in the rotating coordinates which allows a direct application of the standard time-invariant stability criterion to be applied in order to obtain the stability limits. In line boring processes, the depth of cut is usually around the size of corner radius. Li [29] developed a lumped parameter model and incorporated the effect of corner radius. It is assumed that there is only one dominant mode in each of the principal directions. The stability limits for rotating and stationary bar boring were compared and experimental observations provided to support the analytical results. It is shown that there is a significant difference in the stability limits for stationary and rotating bar boring.

In addition Li [29], following the approach of Tsao et al. [11], obtained the equations of motion using the angle of rotation, instead of time, as the independent variable. This angle-domain model is discretized and the spectral radius method is applied to show that modulating the spindle speed leads to larger stability boundaries.

In this chapter, following the work of Sri Namachchivaya and Van Roessel [17], we clarify the mechanism of chatter suppression and enlargement of stability boundaries using SSSV in a boring process.

2.1 Equations of Motion

2.1.1 Rotating Shaft

In deriving the equations of motion, the tool is assumed to be much more flexible in comparison with the machine body and the workpiece. Hence, the tool vibration alone dictates the relative tool-work displacement. Further, in this work, a single tooth is considered and it is also assumed that there is only one dominant mode in each of the principal axes of the tool. Since the axial dynamics are assumed to be negligible in comparison with the radial and tangential dynamics, we consider a two-dimensional lumped mass model for

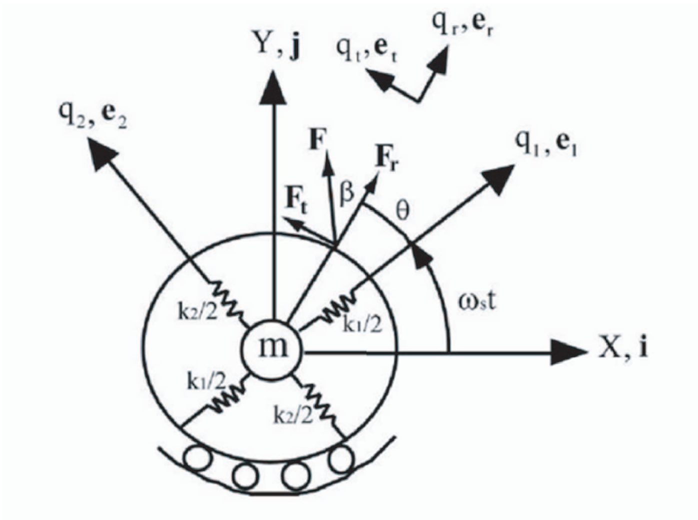

Figure 2.1: Lumped mass model for the boring process (source: Li)

the boring bar as shown in figure 2.1. The tool tip displacements in the radial and tangential directions are denoted by $(q_r,\ q_t)$ while the displacements in the rotational and stationary frames are respectively denoted by $(q_1,\ q_2)$ and $(X,\ Y)$. F_r and F_t are the magnitudes of the radial and tangential forces acting on the tool, respectively. Radial and tangential coordinates $(q_r,\ q_t)$ are oriented at a constant angle α (θ in figure 2.1) relative to the rotational coordinates. $\omega_s(t)$ is the rate of rotation of spindle and is time varying. The magnitude of force F depends on $q_r(t)-\mu_0 q_r(t-\tau)$, where τ is time taken for one revolution, and μ_0 is the overlap factor which scales the effect of $q_r(t-\tau)$ on the uncut chip area. In this study, we focus on the regenerative effect and do not consider the velocity dependent and mode coupling effects. The following model is used for the force:

$$\begin{aligned} f &= -k\,w\,[q_r(t) - \mu_o q_r(t-\tau) + \hat{\beta}_2(q_r(t) - \mu_o q_r(t-\tau))^2 \\ &+ \hat{\beta}_3(q_r(t) - \mu_o q_r(t-\tau))^3] \end{aligned}$$

where

$$q_r(t) = \cos\alpha\ q_1(t) + \sin\alpha\ q_2(t)$$

k_c is the cutting force coefficient, w is the depth of cut, and $\hat{\beta}_2$ and $\hat{\beta}_3$ are the nonlinear cutting force coefficients which are determined from experimental data.

The following equations can be derived.

$$\begin{aligned}\ddot{q}(t) + 2G(t)\dot{q}(t) + D\dot{q}(t) + \dot{G}(t)q(t) + K(t)q(t) + K^d q(t-\tau) \\ = -f_2(q(t), q(t-\tau)) - f_3(q(t), q(t-\tau)), \qquad q \in \mathbb{R}^2\end{aligned} \tag{2.1}$$

where

$$G(t) = \begin{bmatrix} 0 & -\omega_s(t) \\ \omega_s(t) & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2\zeta_1\omega_1 & 0 \\ 0 & 2\zeta_2\omega_2 \end{bmatrix},$$

$$\begin{aligned}
K(t) &= \begin{bmatrix} \omega_1^2 - \omega_s^2(t) + \kappa\cos\alpha\cos(\alpha+\beta) & \kappa\sin\alpha\cos(\alpha+\beta) - 2\zeta_1\omega_1\omega_s \\ \kappa\cos\alpha\sin(\alpha+\beta) + 2\zeta_2\omega_2\omega_s & \omega_2^2 - \omega_s^2(t) + \kappa\sin\alpha\sin(\alpha+\beta) \end{bmatrix} \\
K^d &= -\mu_o\kappa \begin{bmatrix} \cos\alpha\cos(\alpha+\beta) & \sin\alpha\cos(\alpha+\beta) \\ \cos\alpha\sin(\alpha+\beta) & \sin\alpha\sin(\alpha+\beta) \end{bmatrix} \\
\zeta_1 &= \frac{c_1}{2m\omega_1}, \qquad \zeta_2 = \frac{c_2}{2m\omega_2}, \qquad \omega_1^2 = \frac{k_1}{m}, \qquad \omega_2^2 = \frac{k_2}{m}, \qquad \kappa = \frac{kw}{m}
\end{aligned}$$

and f_2, f_3 are repectively the quadratic and cubic nonlinear terms:

$$q_r \stackrel{\text{def}}{=} q_1\cos\alpha + q_2\sin\alpha$$

$$f^2(q(t), q(t-\tau)) = \begin{bmatrix} \frac{1}{m}(k_{20}^{(1)} q_1^2(t) + k_{11}^{(1)} q_1(t)q_2(t) + k_{02}^{(1)} q_2^2(t)) \\ +\kappa\,\hat{\beta}_2(q_r(t) - \mu_o q_r(t-\tau))^2 \cos(\alpha+\beta) \\ \frac{1}{m}(k_{20}^{(2)} q_1^2(t) + k_{11}^{(2)} q_1(t)q_2(t) + k_{02}^{(2)} q_2^2(t)) \\ +\kappa\,\hat{\beta}_2(q_r(t) - \mu_o q_r(t-\tau))^2 \sin(\alpha+\beta) \end{bmatrix}$$

$$f^3(q(t), q(t-\tau)) = \begin{bmatrix} \frac{1}{m}(k_{30}^{(1)} q_1^3(t) + k_{21}^{(1)} q_1^2(t)q_2(t) + k_{12}^{(1)} q_1(t)q_2^2(t) + k_{03}^{(1)} q_2^3(t)) \\ +\kappa\,\hat{\beta}_3(q_r(t) - \mu_o q_r(t-\tau))^3 \cos(\alpha+\beta) \\ \frac{1}{m}(k_{30}^{(2)} q_1^3(t) + k_{21}^{(2)} q_1^2(t)q_2(t) + k_{12}^{(2)} q_1(t)q_2^2(t) + k_{03}^{(2)} q_2^3(t)) \\ +\kappa\,\hat{\beta}_3(q_r(t) - \mu_o q_r(t-\tau))^3 \sin(\alpha+\beta) \end{bmatrix}$$

2.1.2 Periodic spindle speed variation

Since the spindle speed is being modulated continuously, we write

$$\omega_s(t) = \omega_0 + \varepsilon\hat{\sigma}(t), \qquad |\varepsilon| << 1$$

where $\hat{\sigma}(t)$ is a periodic function with amplitude $\hat{\mu}$ and frequency $\hat{\nu}$,with appropriate initial conditions to generate the desired periodic fluctuation.

2.1.3 Augmented autonomous equations

Rescaling time with $s = \omega_1 t$ and defining $u_i(s) = q_i(\frac{s}{\omega_1}) = q_i(t)$, we have $u_i(s - \omega_1\tau) = q_i(t - \tau)$. Now, define $\sigma(s) = \frac{1}{\omega_0}\hat{\sigma}(t)$. Then $\sigma(s)$ is sinusoidal with frequency $\nu = \frac{\hat{\nu}}{\omega_1}$ and amplitude $\mu = \frac{\hat{\mu}}{\omega_0}$.

$$\tau = \frac{2\pi}{\omega_s} = \frac{2\pi}{\omega_0\left[1 + \varepsilon\frac{\hat{\sigma}(t)}{\omega_0}\right]} = \tau_0\left[1 - \varepsilon\sigma(s) + \varepsilon^2\sigma(s)^2 + \ldots\right] \tag{2.2}$$

We use the following notation.

$$\eta_{i,j} = -\frac{1}{\mu_0\kappa\omega_1^2}K^d_{i,j} \qquad \omega_r = \frac{\omega_0}{\omega_1} \qquad \varpi = \frac{\omega_2}{\omega_1} \qquad r = \omega_1\tau_0 \tag{2.3}$$

We then set

$$\begin{aligned} x_1(t) &= u_1(s), \qquad x_2(t) = \dot{u}_1(s) \\ x_3(t) &= u_2(s), \qquad x_4(t) = \dot{u}_2(s) \\ x_5(t) &= -\frac{\dot{\sigma}(s)}{\nu}, \qquad x_6(t) = \sigma(s) \end{aligned} \tag{2.4}$$

The equations of motion may now be written in a form, wherein the explicit time dependent delay terms are replaced by state-dependent delay terms: for example, after such replacement, expression for $\dot{x_2}(t)$ would contain the term

$$x_1\left(t - r(1 - \epsilon x_6(t) + \epsilon^2 x_6^2(t) - \ldots)\right). \tag{2.5}$$

Since the motivating application for this study, machine tool chatter, is known to be a Hopf bifurcation phenomenon, the natural question here is, how do we apply or extend the results from general Hopf bifurcation theories for constant delay equations to the state dependent delay system knowing that the fluctuations in the delay are small. Since the fluctuations are small, $|\varepsilon| << 1$, bounded, and independent of the tool motion, one possible strategy is to, as in [17], use Taylor expansions and expand in powers of $|\varepsilon|$ about a *finite mean delay* r as

$$\begin{aligned} x_1\left(t - r(1 - \varepsilon x_6(t) + \varepsilon^2 x_6^2(t)\right) = x_1(t-r) + \varepsilon r x_2(t-r) x_6(t) + \\ + \varepsilon^2 \left[-r x_2(t-r) x_6^2(t) + \frac{1}{2} r^2 \dot{x}_2(t-r) x_6^2(t) \right] + \ldots \end{aligned} \tag{2.6}$$

Since the original DDE is second order, clearly, $\dot{x}_2(t-r) = \ddot{x}_1(t-r)$ and $\dot{x}_4(t-r) = \ddot{x}_3(t-r)$ are bounded. The problem could be with the higher orders derivatives of $x_1(t-r)$ and $x_3(t-r)$ which may not exist at points kr. In bifurcation studies such as the one that is presented in this paper, we are interested in the asymptotic or long time behavior. Using the method of steps, it is easily shown (see [30], pages 6 – 10) that solutions for nonlinear DDEs with continuously varying delay become smoother with increasing values of time, provided the nonlinear functions are sufficiently smooth. The nonlinear function in the equation governing the system under consideration is C^∞, being a polynomial in $x(t)$ and $x(t - r(t))$, and so the existence of higher order derivatives of the solution is ensured for sufficiently large time t. Though we do not rigorously justify the existence of derivatives beyond second order, due to this smoothing property, we shall assume that sufficient higher derivatives exist while analyzing its asymptotic behavior.

Next, we follow the procedure of order reduction that is often used in the evaluation of the higher time-derivative terms in Eq.(2.6). This procedure uses repeated substitution of the equations of motion to yield a second order equation. For example, say, we have the following equations for $\dot{x}_1(t)$ and $\dot{x}_2(t)$:

$$\dot{x}_1(t) = \wp_1\left(x_1(t), x_1(t-r), x_2(t), x_2(t-r), \dot{x}_2(t-r)\right) \tag{2.7}$$

$$\dot{x}_2(t) = \wp_2\left(x_1(t), x_1(t-r), x_2(t), x_2(t-r)\right) \tag{2.8}$$

Then, by adding a time lag in Eq.(2.8), we replace $\dot{x}_2(t-r)$ on the right hand side of Eq.(2.7) by

$$\dot{x}_2(t-r) = \wp_2\left(x_1(t-r), x_1(t-2r), x_2(t-r), x_2(t-2r)\right)$$

Now that we have given an heuristic justification for the Taylor expansion of the delay terms about a finite mean delay r, we consider the truncated system neglecting the terms of order higher than ε^2. Then we shall drop ε, because we could as well redefine $x_5(t) = \varepsilon x_5(t)$, and $x_6(t) = \varepsilon x_6(t)$ without changing the resulting equations. After performing the taylor expansions, truncating and order reduction, the resulting equations can be concisely written as

$$\dot{x}(t) = E(\kappa)x(t) + D(\kappa)x(t-r) + f(x(t), x(t-r), x(t-2r), \kappa) \tag{2.9}$$

where

$$\begin{aligned}
E(\kappa) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -(1-\omega_r^2+\kappa\eta_{11}) & -2\zeta_1 & -(\kappa\eta_{12}-2\zeta_1\omega_r) & 2\omega_r & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -(\kappa\eta_{21}+2\zeta_2\varpi\omega_r) & -2\omega_r & -(\varpi^2-\omega_r^2+\kappa\eta_{22}) & -2\zeta_2\varpi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 & -\nu & 0 \end{bmatrix} \\
D(\kappa) &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_o\kappa\eta_{11} & 0 & \mu_o\kappa\eta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_o\kappa\eta_{21} & 0 & \mu_0\kappa\eta_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{2.10}$$

and $f = [0\ f_2^1\ 0\ f_2^2\ 0\ 0]^T + [0\ f_3^1\ 0\ f_3^2\ 0\ 0]^T$. f_2^l represents quadaratic nonlinearities and f_3^l represents cubic nonlinearities. $f_2^l = \tilde{f}_2^l + \hat{f}_2^l$ and $f_3^l = \tilde{f}_3^l + \hat{f}_3^l$ where

$$\tilde{f}_2^l = \sum_{j=1}^{4} \xi_j^l x_j(t)x_6(t) + \sum_{j=1}^{4} \xi_j^{l\,\tau} x_j(t-r)x_6(t) + (-1)^l \nu\omega_r x_{5-2l}(t)x_5(t)$$

$$\tilde{f}_3^l = \sum_{j=1}^{4} \gamma_j^l x_j(t)x_6^2(t) + \sum_{j=1}^{4} \gamma_j^{l\,\tau} x_j(t-r)x_6^2(t) + \sum_{j=1}^{4} \gamma_j^{l\,2\tau} x_j(t-2r)x_6^2(t)$$

The separation of nonlinearites $f_i^l = \tilde{f}_i^l + \hat{f}_i^l$ is motivated by the fact that the terms represented by $\tilde{f}$ arise from expansion of delay terms and the terms represented by $\hat{f}$ arise from the nonlinearities originally present in the system (given by f_2 and f_3 in Eq.(2.1)). The coefficients ξ_j^l, $\xi_j^{l\,\tau}$, γ_j^l, $\gamma_j^{l\,\tau}$, $\gamma_j^{l\,2\tau}$ and the terms $\hat{f}_i^l$ are given in the appendix.

The above approximation is quite different from the Taylor expansion in powers of a small constant delay that is discussed on page 243 of [30]. Firstly, we do not expand in a small constant delay about the current time t, but rather we expand in a parameter ε representing the small amplitude of fluctuation about a finite constant delay. Our perturbation expansion results in the original infinite dimensional *state-dependent* DDE being approximated by a DDE with constant delays, along with an augmented system of ODE's to represent the fluctuations. Secondly, we avoid the singular nature of higher order derivatives by using a method of order reduction that is often used to stave off the appearance of higher order derivatives.

Substituting $x(t) = e^{st}x(0)$ in Eq.(2.9) linearized about the trivial solution, we obtain characteristic equation in the following form:

$$\left(s^2 + \nu^2\right) \mathcal{P}(s, e^{-sr}) = 0 \tag{2.11}$$

where $\mathcal{P}$ is a polynomial in the specified arguments, particularly,

$$\mathcal{P}(x, y) = \sum_{i=1}^{4} \sum_{j=1}^{2} p_{rs} x^i y^j \tag{2.12}$$

Clearly, the roots $s = \pm i\nu$ are the eigenvalues associated with the augmented oscillator. When the cutting stiffness parameter equals a critical value, $\kappa = \kappa_c$, the equation $\mathcal{P}(s, e^{-sr}) = 0$ has a pair of purely imaginary roots, $s = \pm i\omega_c$ with the remaining roots having negative real parts.

In the absence of periodic variations in the delay, $\sigma(t) = 0$, the DDE Eq.(2.9) exhibits a Hopf bifurcation at $\kappa = \kappa_c$, with a simple pair of pure imaginary eigenvalues $\pm i\omega_c$ and all the other roots of the characteristic equation have negative real parts. Further, we assume that, in the presence of periodic perturbation, we have two non-resonant pairs of simple eigenvalues $\pm i\omega_c$ and $\pm i\nu$ on the imaginary axis, with $0 < \nu << 1$, such that there is no rational number $\frac{k_1}{k_2}$ with small $|k|$ ($|k| < 5$) satisfying $k_1\omega_c = k_2\nu$, i.e., there are no *primary resonances* due to the low value of ν. However, there are infinitely many *weaker resonances* (with a larger norm $|k|$) which can be neglected due to the presence of dissipation.

In the subsequent sections, we examine the effects of periodic variations in the delay ($\sigma(t) \neq 0$) on the asymptotic stability of the trivial solution of Eq.(2.9) and the associated bifurcations close to the critical parameter, $\kappa = \kappa_c$. We hope that stabilization or further destabilization of the trivial solution for $\kappa > \kappa_c$ may explain the mechanism of SSV in chatter suppression.

2.2 Problem formulation in an FDE framework

The theory of FDE [31, 19, 32, 33] has been developed to a point of high sophistication and provides successful description of the evolution of DDE. For any $r \geq 0$, denote the Banach space of continuous functions from the interval $[-r, 0]$ to $\mathbb{R}^6$ and endowed with sup norm by $\mathcal{C} \stackrel{\text{def}}{=} C([-r, 0], \mathbb{R}^6)$. If $x \in C([-r, \infty), \mathbb{R}^6)$, then for any $t \in [0, \infty)$, we let $x_t \in \mathcal{C}$ be defined by

$$x_t(\theta) \stackrel{\text{def}}{=} x(t + \theta), \quad -r \leq \theta \leq 0$$

to denote a segment of the solution. For each different t, we get a new continuous function x_t on this Banach space. Hence the delay differential Eq.(2.9) is defined as in Hale and Verduyn-Lunel [19]

$$\dot{x}(t) = L(\kappa)x_t + F(x_t, \kappa), \qquad x_0 = \phi \in \mathcal{C}, \tag{2.13}$$

where $L(\cdot)$ is a bounded linear operator from $\mathcal{C} \times \mathbb{R}$ to $\mathbb{R}^6$ and, by the Riesz theorem, it has an unambiguous representation given by the following Stieltjes integral

$$L(\kappa)x_t = \int_{-r}^{0} [d\eta(\theta, \kappa)]x_t(\theta) \tag{2.14}$$

and $F : \mathcal{C} \times \mathbb{R} \to \mathbb{R}^6$ is a smooth nonlinear vector functional. For discrete delays the measure $[d\eta(\theta, \kappa)]$ which defines the linear operator on $\mathcal{C}$ is just a combination of *Dirac delta* functions, that is

$$d\eta(\theta, \kappa) \stackrel{\text{def}}{=} E(\kappa)\delta(\theta)d\theta + D(\kappa)\delta(\theta + r)d\theta \tag{2.15}$$

The characteristic matrix of $L(\kappa)$ is given as

$$\triangle(s,\kappa) \stackrel{\text{def}}{=} sI - \int_{-r}^{0} e^{s\theta}[d\eta(\theta,\kappa)]. \tag{2.16}$$

Setting the determinant of the characteristic matrix equal to zero gives us a transcendental equation which has infinitely many solutions; hence we have an infinite dimensional dynamical system.

For any initial condition $\phi \in \mathcal{C}$, the solution of the DDE is a continuously differentiable function x_t that satisfies Eq.(2.13) for every $t \geq 0$ and $x_0(\theta) = \phi(\theta)$ for every $\theta \in [-r, 0]$. An orbit of a solution is traced out by the family of functions x_t for $t \in [0, \infty)$ (for details, see [19]). It is well known that the translation along the solution of the linear equation

$$\dot{x}(t) = L(\kappa)x_t, \qquad x_0 = \phi \in \mathcal{C} \tag{2.17}$$

induces a strongly continuous semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ defined by the relation

$$T(t)\phi \stackrel{\text{def}}{=} x_t(.;\phi). \tag{2.18}$$

The infinitesimal generator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{C}$ for the strongly continuous semigroup $T(t)$ is given by

$$\begin{aligned} \mathcal{A}\phi &\stackrel{\text{def}}{=} \frac{d\phi(\theta)}{d\theta} \\ \mathcal{D}(\mathcal{A}) &\stackrel{\text{def}}{=} \left\{\phi \in \mathcal{C} : \frac{d\phi}{d\theta} \in \mathcal{C}, \ \frac{d\phi(0)}{d\theta} = L\phi \stackrel{\text{def}}{=} \int_{-r}^{0} [d\eta(\theta,\kappa)]\phi(\theta)\right\} \end{aligned} \tag{2.19}$$

which has only a point spectrum

$$\sigma(\mathcal{A}(\kappa)) = \sigma_P(\mathcal{A}(\kappa)) \stackrel{\text{def}}{=} \{\lambda : \det(\triangle(\lambda,\kappa)) = 0\} \tag{2.20}$$

Suppose that

$$\Lambda \stackrel{\text{def}}{=} \{\lambda \in \sigma(\mathcal{A}(0)) : \Re(\lambda) = 0\} = \{\pm i\omega_c, \pm i\nu\} \tag{2.21}$$

Define P to be the generalized eigenspace associated with the eigenvalues of

Λ with the basis given by Φ

$$\begin{aligned}
\Phi(\theta) &= \begin{bmatrix} e^{i\omega_c\theta} & e^{-i\omega_c\theta} & 0 & 0 \\ i\omega_c e^{i\omega_c\theta} & -i\omega_c e^{-i\omega_c\theta} & 0 & 0 \\ \Gamma\, e^{i\omega_c\theta} & \overline{\Gamma}\, e^{-i\omega_c\theta} & 0 & 0 \\ i\omega_c\Gamma\, e^{i\omega_c\theta} & -i\omega_c\overline{\Gamma}\, e^{-i\omega_c\theta} & 0 & 0 \\ 0 & 0 & e^{i\nu\theta} & e^{-i\nu\theta} \\ 0 & 0 & ie^{i\nu\theta} & -ie^{-i\nu\theta} \end{bmatrix} \\
\Gamma &= -\frac{2i\omega_c\omega_r + 2\zeta_2\varpi\omega_r + \kappa_c\eta_{21}(1-\mu_o e^{-i\omega_c r})}{\varpi^2 - \omega_r^2 + \kappa_c\eta_{22}(1-\mu_o e^{-i\omega_c r}) - \omega_c^2 + i2\zeta_2\varpi\omega_c}
\end{aligned} \tag{2.22}$$

The following Ψ may be choosen as a basis for the generalized eigenspace (dual space P^*) of the transposed equation associated with Λ

$$\Psi(\tau) = \begin{bmatrix} e^{-i\omega_c\tau} & e^{i\omega_c\tau} & 0 & 0 \\ \Gamma_1\, e^{-i\omega_c\tau} & \overline{\Gamma}_1\, e^{i\omega_c\tau} & 0 & 0 \\ \Gamma_2\, e^{-i\omega_c\tau} & \overline{\Gamma}_2\, e^{i\omega_c\tau} & 0 & 0 \\ \Gamma_3\, e^{-i\omega_c\tau} & \overline{\Gamma}_3\, e^{i\omega_c\tau} & 0 & 0 \\ 0 & 0 & \frac{1}{2}e^{-i\nu\tau} & \frac{1}{2}e^{i\nu\tau} \\ 0 & 0 & -\frac{i}{2}e^{-i\nu\tau} & \frac{i}{2}e^{i\nu\tau} \end{bmatrix}^T$$

$$\begin{aligned}
\Gamma_1 &= \frac{2i\omega_c\omega_r + 2\zeta_2\varpi\omega_r + \kappa_c\eta_{21}(1-\mu_o e^{-i\omega_c r})}{\Gamma_1^{den}} \\
\Gamma_1^{den} &= -2\omega_r\left(1 - \omega_r^2 + \kappa_c\eta_{11}(1-\mu_o e^{-i\omega_c r})\right) \\
&+ \left(i\omega_c + 2\zeta_1\right)\left(2\zeta_2\varpi\omega_r + \kappa_c\eta_{21}(1-\mu_o e^{-i\omega_c r})\right) \\
\Gamma_2 &= \frac{i\omega_c + 2\zeta_2\varpi}{2\omega_r} - \Gamma_1\left(\frac{(i\omega_c + 2\zeta_1)(i\omega_c + 2\zeta_2\varpi)}{2\omega_r} + 2\omega_r\right) \\
\Gamma_3 &= \frac{1}{2\omega_r} - \Gamma_1\frac{i\omega_c + 2\zeta_1}{2\omega_r}
\end{aligned} \tag{2.23}$$

The basis Ψ would be normalized by the condition $\langle\Psi, \Phi\rangle = I$, so that further calculations could be simplified. The bilinear form $\langle\cdot,\cdot\rangle$ is given by

$$\langle\psi, \phi\rangle = (\psi(0), \phi(0)) - \int_{-r}^{0}\int_{0}^{\theta}\psi(\xi-\theta)[d\eta(\theta,\kappa)]\phi(\xi)d\xi \tag{2.24}$$

and $(\cdot,\cdot)$ stands for Hermite inner product.

Then $\mathcal{C}$ can be decomposed by Λ as

$$\mathcal{C} = P \oplus Q, \quad \text{where} \quad Q \stackrel{\text{def}}{=} \{\phi \in \mathcal{C} : \langle \Psi, \phi \rangle = 0\} \tag{2.25}$$

and any element $x_t \in \mathcal{C}$ can be written as $x_t = x_t^P + x_t^Q$ where $x_t^P \in P$ with

$$x_t^P = \Phi \langle \Psi, x_t \rangle \quad \text{and} \quad x_t^Q \in Q$$

2.2.1 Nonlinear problem

Making use of $T(t)$, in the integrated form, Eq.(2.13) with initial data ϕ becomes

$$x_t = T(t)\phi + \int_0^t T(t-s) X_0 F(x_s, \kappa) ds, \tag{2.26}$$

where

$$X_0 = X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0, \\ I, & \theta = 0. \end{cases}$$

Equation Eq.(2.26) on differentiation with respect to time yields a formal expression

$$\frac{dx_t}{dt} = \mathcal{A}(\kappa)x_t + X_0 F(x_t, \kappa), \qquad x_0 = \phi \in \mathcal{C}. \tag{2.27}$$

Although the solution space of the abstract linear equation is $\mathcal{C}$, the nonlinear Eq.(2.27) has jump discontinuities at $\theta = 0$. Hence, as pointed out by [20], the appropriate solution space for Eq.(2.27) is

$$\begin{aligned} \mathcal{BC} &\stackrel{\text{def}}{=} \mathcal{C} \oplus < X_0 > \\ &= \left\{ \phi : [-r, 0] \to \mathbb{R}^6 ; \phi \text{ is continuous on } [-\text{r}, 0) \text{ with jump discontinuities at } 0 \right\} \end{aligned}$$

However, $\mathcal{A}$ maps elements only to $\mathcal{C}$ and to remedy this problem we define, as in Chow and Mallet-Paret [20], an extension of $\mathcal{A}$ denoted by

$$\hat{\mathcal{A}}\phi \stackrel{\text{def}}{=} \phi' + X_0 \{L\phi - \phi'(0)\} \tag{2.28}$$

which takes values not just into $\mathcal{C}$ but into $\mathcal{BC}$. Now the domain, $\mathcal{D}(\hat{\mathcal{A}})$, is extended to all of $\mathcal{C}^1 \stackrel{\text{def}}{=} \{\phi \in \mathcal{C} : \phi' \in \mathcal{C}\}$. Then any solution of Eq.(2.9) for

$t \geq 0$, satisfies the abstract ODE in $\mathcal{BC}$

$$\frac{dx_t}{dt} = \hat{\mathcal{A}}(\kappa)x_t + X_0 F(x_t, \kappa), \qquad x_0 = \phi \in \mathcal{C}. \tag{2.29}$$

This procedure of treating the equation in $\mathcal{BC}$ was first introduced by [20] and was used effectively by Faria and Magalhaes [34] in their development of normal form theory for **FDE**. Let the projection $\hat{\phi}^P$ of any $\hat{\phi} \in \mathcal{BC}$ onto P be defined as

$$\pi : \mathcal{BC} \to P, \qquad \pi\left(\phi + X_0\beta\right) \stackrel{\text{def}}{=} \Phi\left[\langle \Psi, \phi \rangle + \Psi(0)\beta\right]. \tag{2.30}$$

It can be shown [34] that the projection π commutes with $\hat{\mathcal{A}}$ in $\mathcal{C}^1$, that is, $\hat{\mathcal{A}}\pi = \pi\hat{\mathcal{A}}$ for elements in $\mathcal{C}^1$. Making use of this commutative property of π and the decomposition of $\mathcal{C} = P \oplus Q$ it can be easily shown that $\mathcal{BC}$ has a direct sum decomposition, that is

$$\mathcal{BC} = P \oplus \ker(\pi).$$

Then, for $x_t \in \mathcal{C}^1$, we write

$$x_t = \Phi z(t) + y_t, \quad \text{where} \quad z \in \mathbb{R}^4 \;\text{ and }\; y_t \in Q \cap \mathcal{C}^1. \tag{2.31}$$

Making use of the domain decomposition Eq.(2.31), the relation $\mathcal{A}\Phi = \Phi\hat{B}$ where

$$\hat{B} = \begin{bmatrix} i\omega_c & 0 & 0 & 0 \\ 0 & -i\omega_c & 0 & 0 \\ 0 & 0 & i\,\nu & 0 \\ 0 & 0 & 0 & -i\,\nu \end{bmatrix} \tag{2.32}$$

and the fact that $\langle \Psi, y_t \rangle = 0$, in the abstract ODE Eq.(2.29) yields

$$\begin{aligned} \Phi\dot{z}(t) + \frac{dy_t}{dt} &= \Phi\hat{B}z(t) + (I-\pi)\hat{\mathcal{A}}y_t \\ &\quad + \Phi\Psi(0)F\left(\Phi z(t) + y_t, \kappa\right) + (I-\pi)X_0F\left(\Phi z(t) + y_t, \kappa\right). \end{aligned} \tag{2.33}$$

Now projecting Eq.(3.19) onto P and its complement in $\mathcal{BC}$ yields

$$\begin{aligned}\dot{z}(t) &= \hat{B}z(t) + \Psi(0)F\left(\Phi z(t) + y_t, \kappa\right) \\ \frac{dy_t}{dt} &= (I-\pi)\hat{\mathcal{A}}y_t + (I-\pi)X_0F\left(\Phi z(t) + y_t, \kappa\right)\end{aligned} \tag{2.34}$$

Hence, the abstract ODE Eq.(2.29) in $\mathcal{BC}$ is equivalent to Eq.(3.20), and it is very important to realize that these *almost decoupled equations* are the *starting point* for the rest of our analysis. The second equation in Eq.(3.20) is interpreted as an equality for each $\theta \in [-r, 0]$, but we may informally think of it as an equation in $Q \cap \mathcal{C}^1$. The complete decomposition of Eq.(3.20) into a *four-dimensional equation* and an infinite-dimensional equation is the main goal of the subsequent section.

2.3 Center manifold reduction

In this section we compute the normal forms in the four dimensional locally invariant center manifold. The essential dynamic behavior of the infinite dimensional system Eq.(2.13) is determined by the evolution of a subset of the possible modes, the "critical" modes, Φz. When many of the modes are "heavily damped", trajectories are rapidly attracted to some low-dimensional invariant manifold, which may be parameterized by the amplitudes, z, of the critical modes. This geometric picture is at the heart of the application of center manifold reduction to the rational construction of low-dimensional models. We now briefly summarize the results on center manifold reduction for FDE and apply them to compute the normal forms for the non-resonant Hopf bifurcation of the trivial equilibrium of Eq.(2.13). The existence of such a manifold for FDE was first proved by Chafee [35] and is given by

$$W^c_{loc}(0) = \{\phi \in \mathcal{C} : \phi = \Phi z + h(z),\ z \in V\}, \tag{2.35}$$

where V is a neighborhood of zero in $\mathbb{R}^4$. The center manifold theorem assures that the four-dimensional invariant manifold is tangent to the center subspace P, that is $h(0) = 0$ and $D_z h(0) = 0$, and $h : V \to Q \cap \mathcal{C}^1$ is C^k-smooth. Furthermore, the long term behavior of solutions of the original

DDE Eq.(2.13) is described by the solutions of the four-dimensional ODE

$$\dot{z}(t) = \hat{B}z(t) + \Psi(0)F\left(\Phi z(t) + h\left(z(t)\right), \kappa_c\right). \tag{2.36}$$

In Eq.(2.36), $\hat{B}$ is the 4×4 diagonal matrix of eigenvalues with zero real part, $\Psi(0)$ is the basis evaluated at $\theta = 0$ for the invariant dual subspace P^*, and Φ is the basis for the invariant subspace P given, respectively, by equations Eq.(2.32), Eq.(2.23) and Eq.(2.22). The stability on the center manifold decides the stability of the original equation. This is the framework in which we shall construct the center manifold and then the normal forms to study the bifurcations of the trivial equilibrium in Eq.(2.13).

2.3.1 Construction of the center manifold

Construction of center manifolds for FDE is still a computationally intensive exercise, unlike their construction for ODEs. Taking the solution of the center manifold as $x_t(\theta) = \Phi(\theta)z(t) + h\left(z(t);\theta\right)$ in Eq.(2.13) or equivalently, substituting for y_t the expression $h\left(z(t);\theta\right)$ in Eq.(3.20) yields

$$\begin{aligned} D_z h\left(z(t);\theta\right)\dot{z}(t) &= \mathcal{A}h\left(z(t);\theta\right) - \Phi(\theta)\Psi(0)F\left(\Phi z(t) + h\left(z(t);\theta\right), \kappa_c\right) \\ &\quad + X_0\left[Lh\left(z(t);\theta\right) - h'\left(z(t);0\right)F\left(\Phi z(t) + h\left(z(t);\theta\right), \kappa_c\right)\right] \end{aligned} \tag{2.37}$$

where $\dot{z}(t)$ is given by the four-dimensional ODE Eq.(2.36), which on substitution in the left hand side of Eq.(2.37) yields a system of partial differential equations for $h\left(z;\theta\right)$, viz.

$$\begin{aligned} &D_z h\left(z;\theta\right)\hat{B}z + D_z h\left(z;\theta\right)\Psi(0)F(\Phi z(t) + h\left(z;\theta\right), \kappa_c) \\ &\qquad = \mathcal{A}h\left(z;\theta\right) - \Phi(\theta)\Psi(0)F(\Phi z(t) + h\left(z;\theta\right), \kappa_c) \quad -r \le \theta < 0, \\ &D_z h\left(z;0\right)\hat{B}z + D_z h\left(z;0\right)\Psi(0)F(\Phi z(t) + h\left(z;0\right), \kappa_c) \\ &\quad = \int_{-r}^{0}[d\eta(\tau,0)]h\left(z;\tau\right) + \left(I - \Phi(0)\Psi(0)\right)F(\Phi z(t) + h\left(z;0\right), \kappa_c) \quad \theta = 0. \end{aligned} \tag{2.38}$$

We can approximate $h\left(z;\theta\right)$, using the standard approach in center manifold theory (see, for example, [36]), as a polynomial or power series in z. Construction of center manifolds and numerical calculation (approximation) of Hopf bifurcation for FDE was given by Hasard et al. [32] for the first time.

Belair and Campbell [37] and [38] have automated the algebraic construction of two-dimensional center manifolds and the subsequent calculations of Hopf bifurcation for FDE.

For non-degenerate Hopf bifurcations, only terms up and including $|z|^3$ are needed in the normal forms and it is sufficient to construct an approximation to the center manifold up to and including $|z|^2$. Hence, using the fact that $h\,(z;\theta)$ is tangent to the center subspace P, we obtain the following approximation

$$h_s\,(z;\theta) = \sum_{|k|=2} w_{s:k}\,(\theta)\,z^k = \sum_{|k|=2} w_{s:k_1k_2k_3k_4}\,(\theta)\,z_1^{k_1}z_2^{k_2}z_3^{k_3}z_4^{k_4}, \quad s = 1, 2, ..., 4 \tag{2.39}$$

which on substitution into Eq.(2.38) results in a set of forty ODEs for $w_{s:k}\,(\theta)$,

$$\begin{aligned} &\sum_{|k|=2}\left[w'_{s:k}\,(\theta) - \sum_{i=1}^{4} k_i\lambda_i w_{s:k}\,(\theta)\right] z^k \\ &\qquad = \Phi_{sj}(\theta)\Psi_{jl}(0)F_l^{(2)}(\Phi z(t)) \qquad -r \leq \theta < 0 \end{aligned} \tag{2.40}$$

where $F_j^{(2)}(\Phi z(t))$ represents the quadratic nonlinear terms evaluated on the manifold coordinates, along with the boundary conditions

$$\begin{aligned} &\sum_{|k|=2}\left[\sum_{i=1}^{4} k_i\lambda_i w_{s:k}\,(0) - \sum_{j=1}^{4} E_{sj}w_{j:k}(0) - \sum_{j=1}^{4} D_{sj}w_{j:k}(-r)\right] z^k \\ &\qquad = \left(\delta_{sj} - \Phi_{sl}(0)\Psi_{lj}(0)\right) F_j^{(2)}(\Phi z(t)) \qquad \theta = 0 \end{aligned} \tag{2.41}$$

It is clear from the above analysis that substituting the results for $h\,(z;\theta)$ into Eq.(2.36) and truncating, yields the following four-dimensional **ODE** on the center manifold

$$\dot{z}_i(t) = \hat{B}_{ij}z_j(t) + \sum_{|k|=2} v_{i:k}z^k(t) + \sum_{|k|=3} u_{i:k}z^k(t), \qquad j = 1, 2, ..., 4, \tag{2.42}$$

where $v_{i:k}$ represents the coefficients of the quadratic nonlinear terms evaluated from $\Psi(0)F(\Phi z(t))$ while $u_{i:k}$ represents both the coefficients of the cubic nonlinear terms evaluated from $\Psi(0)F(\Phi z(t))$ as well as the center manifold corrections given by appropriate $w_{i:k}\,(0)$ and $w_{i:k}\,(-r_0)$. As will be shown in the subsequent section, not all the explicit solutions of $w_{s:k}\,(\theta)$ are needed. Procedure to find those that are necessary would be given in the appendix.

2.4 Computation of the normal form

By a standard nonlinear change of variables Guckenheimer and Holmes [39], the equations on the center manifold, Eq.(2.42), can be brought into a normal form. Based on our assumption, the two pairs of pure imaginary eigenvalues, at $\kappa = \kappa_c$, are such that there are no rational numbers $\frac{k_1}{k_2}$ with small $|k|$ satisfying $k_1\omega_c = k_2\nu$. We now compute the normal forms of the reduced equations (2.42) in complex vector space $\mathbb{C}^2$ (for the ease of computation), under the assumption of no low-order resonances (more specifically no resonances for $|k| < 5$), as

$$\begin{aligned}
\dot{z}_1(t) &= i\omega_c z_1(t) + \left[g_{1:2100} z_1(t) z_2(t) + g_{1:1011} z_3(t) z_4(t)\right] z_1(t) \\
\dot{z}_2(t) &= -i\omega_c z_2(t) + \left[g_{2:1200} z_1(t) z_2(t) + g_{2:0111} z_3(t) z_4(t)\right] z_2(t) \\
\dot{z}_3(t) &= i\nu z_3(t), \qquad \dot{z}_4(t) = -i\nu z_4(t),
\end{aligned} \tag{2.43}$$

where the normal form coefficients $g_{2:1200} = \overline{g_{1:2100}}$ and $g_{2:0111} = \overline{g_{1:1011}}$ contain terms which depend on $w_{i:k}(0)$ and $w_{i:k}(-r)$. The expressions for $g_{1:2100}$ and $g_{1:1011}$ are very involved and hence we illustrate the procedure to find them in appendix and evaluate them numerically. Identifying $z_2 = \bar{z}_1$, $z_4 = \bar{z}_3$, the amplitude of the spindle speed modulation as $\mu = \sqrt{|z_3 z_4|}$ (a constant which depends on the initial conditions of the augmented ODE), and introducing the linear unfolding term given by

$$\lambda' \stackrel{\text{def}}{=} \frac{d\lambda}{d\kappa}\Big|_{\kappa=\kappa_c}$$

the differential equation for the unknown amplitude $z \stackrel{\text{def}}{=} z_1$, is obtained as

$$\dot{z}(t) = i\omega_c\, z(t) + (\kappa - \kappa_c)\,\lambda'\, z(t) + |\mu|^2\, S(r,\nu) z(t) + \Lambda(r)\, |z(t)|^2\, z(t) \tag{2.44}$$

Hence, the main objective of this paper can now be answered by studying the reduced nonlinear equation (2.44), which represents the “normal form” of the original system with periodic time delay.

2.5 Stability and bifurcation analysis

Letting $z(t) = r(t)\exp(i\phi(t))$, we can rewrite Eq.(2.44) in polar coordinates as

$$\begin{aligned} \dot{r}(t) &= \left[(\kappa-\kappa_c)\delta_c + \mathcal{R}(r,\nu)\ |\mu|^2 + \Lambda^{Re}(r) r^2(t)\right]\ r(t) \\ \dot{\phi}(t) &= \omega_c + (\kappa-\kappa_c)\omega_c' + \mathcal{I}(r,\nu)\ |\mu|^2 + \Lambda^{Im}(r)\, r^2(t), \end{aligned} \tag{2.45}$$

where δ_c is the real part of $\lambda'(\kappa_c)$ and represents the rate at which the eigenvalue is crossing the imaginary axis, $\mathcal{R}(r_0,\nu)$ and $\mathcal{I}(r_0,\nu)$ are the real and imaginary parts of $S(r,\nu)$ and $\Lambda^{Re}(r)$ and $\Lambda^{Im}(r)$ are the real and imaginary parts of the nonlinear coefficient $\Lambda(r)$. Since there are no resonances, the bifurcation equations Eq.(2.45) have $\mathbb{S}^1$ symmetry and the phase is decoupled from the amplitude of the nonlinear response.

First, we clarify the mechanism for the suppression of regenerative chatter by examining the stability of Eq.(2.45) linearized about the trivial solution

$$\dot{r}(t) \ = \ \left[(\kappa-\kappa_c)\delta_c \ + |\mu|^2\ \mathcal{R}(r,\nu)\right]\ r(t), \tag{2.46}$$

If $\delta_c > 0$, for $\mu = 0$ (no periodically varying delay), the system is unstable for $\kappa > \kappa_c$. Clearly, when $\mu \neq 0$ stabilization is possible only if $R(r_0,\nu)$ is negative.

From Eq.(2.46), we derive a new stability boundary which depends on the amplitude μ, and frequency ν of the variations in the delay and the bifurcation parameter κ. The stability boundary when the delay is time varying is changed by the amount

$$\kappa_c^{ssv} \ \stackrel{\text{def}}{=} \ -\,|\mu|^2\ \frac{\mathcal{R}(r,\nu)}{\delta_c} \tag{2.47}$$

in comparison with the stability boundary for the constant delay case. Hence, positive values of κ_c^{ssv} imply a stabilization effect while negative values of κ_c^{ssv} imply further destabilization.

From Eq.(2.45), we have

$$r_1 = 0 \quad \text{and} \quad r_2 = \sqrt{-\frac{(\kappa-\kappa_c)\delta + \mathcal{R}(r,\nu)\ |\mu|^2}{\Lambda^{Re}(r)}} \tag{2.48}$$

as the stationary solutions. The nontrivial solution indicates a delayed Hopf bifurcation and the sign of $\Lambda^{Re}(r)$ governs the qualitative behavior close to the new bifurcation point $\kappa = \kappa_c + \kappa_c^{ssv}$.

The bifurcation is supercritical when $\Lambda^{Re}(r) < 0$, and subcritical when $\Lambda^{Re}(r) > 0$. In both cases, the trivial solution becomes unstable for $\kappa > \kappa_c$. However, in subcritical bifurcations the increase of oscillation amplitude is sudden and sometimes very dangerous; a well known result in classical bifurcation theory. Once identified, bifurcation control techniques such as nonlinear velocity feedback (see, for example, [27]) may be used to convert a sub-critical bifurcation into a super-critical one.

2.6 Application to chatter suppression

We study the linearized model of (2.1) i.e. we set the nonlinearities in the RHS of (2.1) to zero. Following Pratt [27], we use the following values for the various parameters in the model.

$$\begin{aligned} \omega_1 &= 365 \text{ Hz}, \qquad \omega_2 = 524 \text{ Hz}, \qquad \zeta_1 = 0.02, \qquad \zeta_2 = 0.03 \\ \alpha &= -15^o, \qquad \beta = 45^o, \qquad \mu_o = 1.0 \end{aligned}$$

All the calculations to be done according to the previous sections are presented in the appendix. Here, we summarize our results in the following figures. We select a spindle speed amplitude modulation of 5% i.e. $\mu = 0.05$ with frequency one-tenth of ω_1 i.e. $\nu = 0.1$.

Stability chart with no SSV is shown in figure 2.2. Region of stability of the trivial solution is indicated in the figure. Last lobe on the right of figure has chatter frequency greater than ω_1. In the range of ω_r shown in the figure, lobes alternate between chatter frequency greater or less than the natural frequency ω_1.

The results of the calculations indicated in sections 2.3, 2.4 and 2.5 are summarized in the figures that follow. Figure 2.3 shows that $\delta_c > 0$.

Figure 2.4 shows the variation of $\mathcal{R}$ with ω_r and it is clear that modulating the spindle speed has a stabilizing effect.

In figure 2.5 we obtain the new enlarged stability boundary.

In figure 2.6 we show numerical simulation of (2.1) with $\omega_r = 0.5$ and κ at the critical value, with no SSV. State dependent delay equation solver *ddesd*

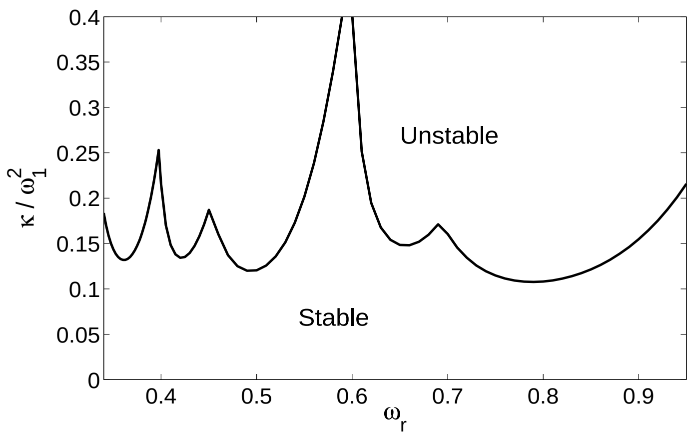

Figure 2.2: Stability Chart

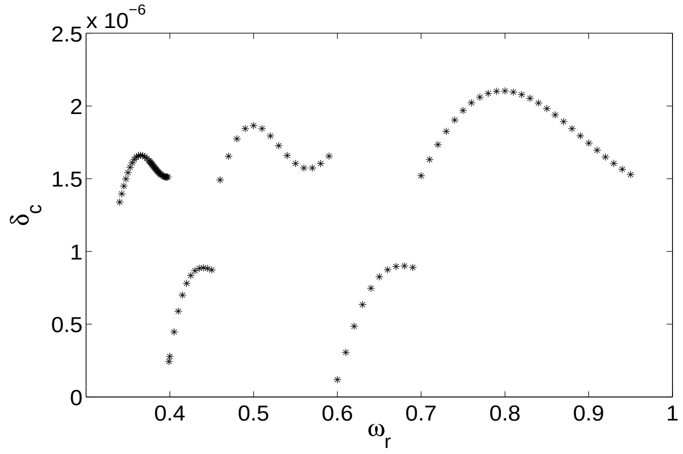

Figure 2.3: Real part of the crossing rate of imaginary eigenvalue

of MATLAB was used for simulation. Oscillations of constant amplitude (eigenvalues are exactly on the imaginary axis) can be seen. With same ω_r and κ we apply a spinlde speed modulation of 5%. Then amplitude of the oscillations decrease. Result is shown in figure 2.7. Note that the horizontal axis in these figures is not the real time but the scaled one. $t = 600$ in the graph corresponds to $\frac{600}{\omega_1} \approx 2$seconds.

However, the enlargement of stability boundary obtained by the analytic means does not match with simulations. For example, at $\omega_r = 0.5$, analytic method gives an increase of 30% in κ. But numerical simulations indicate an increase only of about 12%. In figure 2.8 we show numerical simulation at $\kappa = 1.12\kappa_c$. Decrease in amplitude is insignificant.

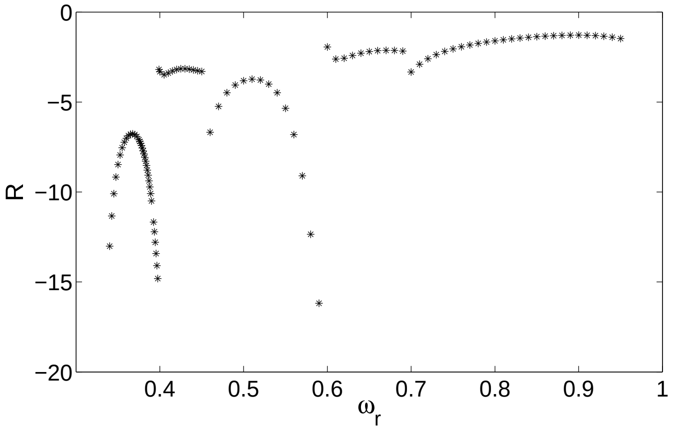

Figure 2.4: variation of $\mathcal{R}$ with frequency of SSV

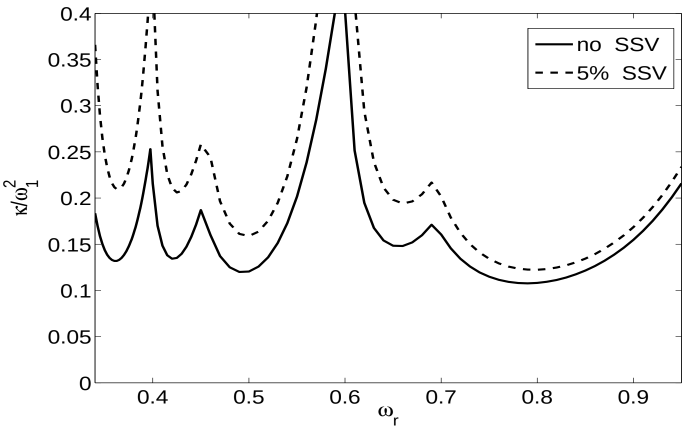

Figure 2.5: enlargement of stability boundary due to SSV

2.7 Conclusion

Mechanism of chatter suppression is explained. It is shown that SSSV technique suppresses chatter. Analytical results for enlargment of stability boundary are obtained. Numerical simulations indicate that actual enlargment obtained is less than that given by analytical results.

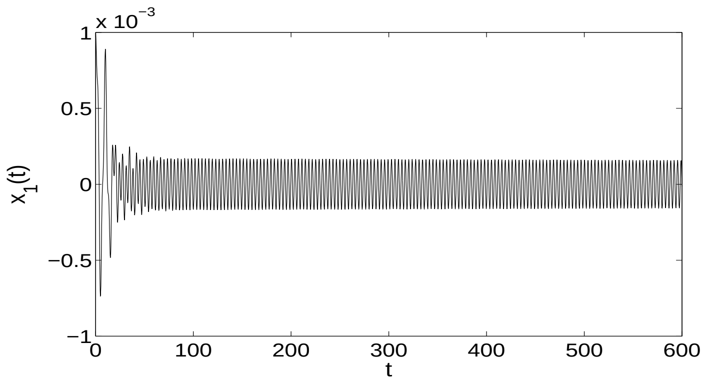

Figure 2.6: On the verge of instability with no SSV

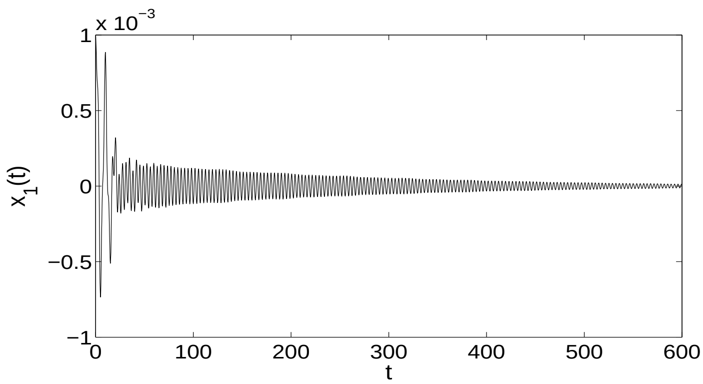

Figure 2.7: application of 5% speed modulation results in chatter suppression

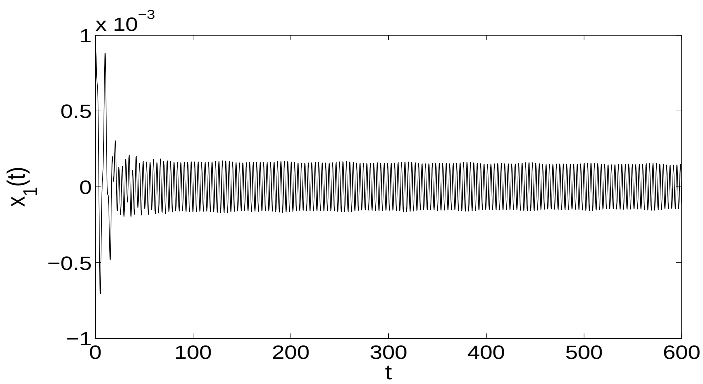

Figure 2.8: numerical simulation with $\omega_r = 0.5$, $\kappa = 1.12\kappa_c$ and 5% speed modulation

CHAPTER 3

CHATTER SUPPRESSION BY VARYING SPINDLE SPEED RANDOMLY

In this chapter, we attempt to answer whether suppression of chatter can be achieved by varying spindle speed randomly. Motivated by the results from Namachchivaya and Wishtutz [1], we consider the spindle speed being modulated according to a two state markov noise. In section 3.1 we present the model of machining used. We formulate the problem in FDE framework in 3.2. Next, in sections 3.3 and 3.4 we study illustrate procedure to find the asymptotic expansion for top Lyapunov exponent. In section 3.5 we calculate explicity the asymptotic expansion to second order and in section 3.6 we answer the question whether chatter suppression can be achieved.

3.1 Hanna-Tobias model and spindle speed variation

Chatter in machining is attributed to Regenerative effect. Note that the surface generated by the tool on the pass becomes the upper surface of the chip on the subsequent pass. Thus, thickness of the chip being cut depends on both the current state and the state one revolution ago. Since the force acting on the tool is a function of the chip being cut, it also depends on the past state, and hence modeling results in Delay Differential Equations. The figure 3.1 (from [17]), illustrates the dynamics. We study the following linear model of Hanna and Tobias.

$$\ddot{q}(t) + 2\zeta p\dot{q}(t) + p^2 q(t) = -\kappa p^2 \left\{ q(t) - q(t-\tau) \right\} \tag{3.1}$$

where p is natural frequency, ζ is damping coefficient and k is width of cut parameter. $\tau = \frac{2\pi}{\omega_s}$, where ω_s is spindle speed. Since the spindle speed is being modulated, we write

$$\omega_s(t) = \omega_0 \left[1 - \varepsilon\sigma(\tilde{\xi}(t)) \right], \qquad |\varepsilon| << 1$$

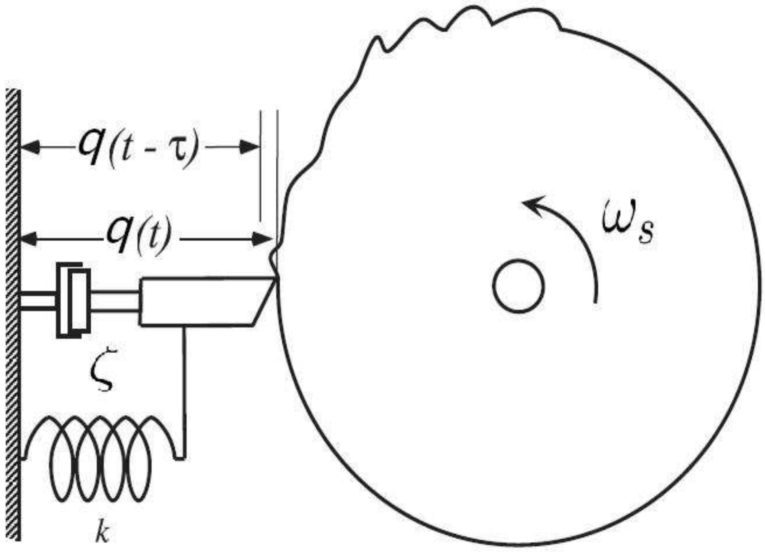

Figure 3.1: Modeling the tool dynamics

where $\tilde{\xi}(t)$ is a noise term and $\sigma(\tilde{\xi})$ is a mean zero function of the noise $\tilde{\xi}$. Rescaling time with $s = pt$ and defining $u(s) = q(\frac{s}{p}) = q(t)$, we have $u(s - p\tau) = q(t - \tau)$. Now, define $\xi(s) = \tilde{\xi}(\frac{s}{p}) = \tilde{\xi}(t)$. The resulting equation is

$$\ddot{u}(s) + 2\zeta\dot{u}(s) + u(s) = -\kappa\left\{u(s) - u(s - r(1 + \varepsilon\sigma(\xi(s)) + \ldots))\right\}$$

where $r = \frac{2\pi p}{\omega_0}$. Define $x_1(t) = u(s)$ and $x_2(t) = \dot{u}(s)$. The above equation can be written (expanding to the order ε) as

$$\dot{x}(t) = Ex(t) + Dx(t - r) + \varepsilon F \tag{3.2}$$

where $x(t) = \{x_1(t), x_2(t)\}^T$ and

$$E = \begin{bmatrix} 0 & 1 \\ -(1+\kappa) & -2\zeta \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ \kappa & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ -\kappa\, r\, x_2(t-r) \end{bmatrix} \sigma(\xi)$$

3.2 Problem Formulation as FDE

We rephrase (3.2) in FDE framework as discussed in (1.1). We denote the Banach space of continuous real-valued functions from the interval $[-r, 0]$ to

$\mathbb{R}^2$ by $\mathcal{C} \stackrel{\text{def}}{=} C([-r,0],\mathbb{R}^2)$ endowed with sup norm. If $x \in C([-r,\infty),\mathbb{R}^2)$, then for any $t \in [0,\infty)$, we let $x_t \in \mathcal{C}$ be defined by

$$x_t(\theta) \stackrel{\text{def}}{=} x(t+\theta), \quad -r \leq \theta \leq 0$$

to denote a segment of the solution. For each different t, we get a new continuous function x_t on this Banach space. Hence following Hale and Verduyn-Lunel, the delay differential equation (3.2) is re-written as

$$\dot{x}(t) = L_0 x_t + \varepsilon f(x_t, \xi(t)), \tag{3.3}$$

where L_0 is a bounded linear operator from $\mathcal{C}$ to $\mathbb{R}^2$ which by the Riesz theorem have representation given by the following Stieltjes integral

$$L_0\phi = \int_{-r}^{0} [E\,\delta(\theta) + D\,\delta(\theta+r)]\, d\theta\, \phi(\theta) \tag{3.4}$$

and $f(x_t,\xi(t))$ can be written as $\xi(t)\, L_1 x_t$, where L_1 is a bounded linear operator from $\mathcal{C}$ to $\mathbb{R}^2$ which by the Riesz theorem have representation given by the following Stieltjes integral

$$L_1\phi = \int_{-r}^{0} [\tilde{D}\,\delta(\theta+r)]\, d\theta\, \phi(\theta) \tag{3.5}$$

where

$$\tilde{D} = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa\, r \end{bmatrix}$$

First, consider the unperturbed problem,

$$\dot{x}(t) = L_0 x_t \tag{3.6}$$

This unperturbed problem, with initial conditions $x_0 = \phi \in \mathcal{C}$ is an autonomous FDE, and has a unique solution which is related to the solution of (3.1), with constant spindle speed, as $x_t(\cdot;\phi) = x(t+\cdot;\phi)$. Translation along the solution induces a strongly continuous semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ defined by the relation

$$T(t)\phi(.) \stackrel{\text{def}}{=} x_t(.;\phi), \qquad t \geq 0, \tag{3.7}$$

where

$$x_t(\theta;\phi) = T(t)\phi(\theta) \stackrel{\text{def}}{=} \begin{cases} \phi(t+\theta), & t+\theta \le 0, \\ \phi(0) + \int_0^{t+\theta} L_0(s)\,(T(s)\phi)\,ds, & t+\theta > 0 \end{cases}$$

The infinitesimal generator $A : \mathcal{D}(A) \to \mathcal{C}$ of the strongly continuous semigroup $T(t)$ is given by

$$\begin{aligned} A\phi &\stackrel{\text{def}}{=} \frac{d\phi(\theta)}{d\theta}, \\ \mathcal{D}(A) &\stackrel{\text{def}}{=} \left\{ \phi \in \mathcal{C}^1 : \frac{d\phi(0)}{d\theta} = L_0(\tau)\phi \right\}, \end{aligned} \tag{3.8}$$

where $\mathcal{C}^1$ is the Banach space $C^1([-r,0],\mathbb{R}^2)$, endowed with the sup norm. If $\phi(\cdot) \in \mathcal{D}(A)$, then so is $x_t(\cdot;\phi)$, and putting

$$\frac{dx_t(\theta)}{dt} = \frac{dx(t+\theta)}{dt}, \quad t = 0$$

we have

$$\frac{dx_t}{dt} = \frac{dT(t)\phi}{dt} = A(\tau)T(t)\phi = A(\tau)x_t, \quad t \ge 0, \tag{3.9}$$

For initial conditions ϕ from $\mathcal{D}(A(\tau))$, equations (3.6) and (3.9) are equivalent in the sense that $x(t), t \ge 0$, solves the **FDE** (3.6) with $x_0 = \phi \in \mathcal{D}(A(\tau))$ iff $x_t(\theta)$, $t \ge 0$, $\theta \in [-r,0]$, solves (3.9) with $x_0 = \phi \in \mathcal{D}(A(\tau))$.

The characteristic equation of L_0 is given as

$$\det(\triangle(\lambda)) = 0 \tag{3.10}$$

where

$$\triangle(\lambda) \stackrel{\text{def}}{=} \lambda I + L_0 e^{\lambda\theta}$$

(3.10) is a transcendental equation which has infinitely many solutions. Spectrum $\sigma(A(\tau))$ of the generator $A(\tau) : \mathcal{D}(A(\tau)) \to \mathcal{C}$ is defined as

$$\sigma(A(\tau)) \stackrel{\text{def}}{=} \{\lambda : \triangle(\lambda) = 0\} \tag{3.11}$$

It is known that the infinitesimal generator has only a point spectrum and for any $\gamma \in \mathbb{R}$ there are only finitely many eigenvalues with real part of λ greater than γ.

Note that, for a given r, the characteristic equation depends on κ. When $\kappa = 0$, there is only one pair of eigenvalues whose real part is negative. For $\kappa > 0$, the equation is transcendental and there are infinitely many eigenvalues. As κ increases, some of the roots may cross the imaginary axis. Suppose at $\kappa = \kappa_c$, first such imaginary axis crossing occurs. Then assuming $\lambda = i\omega_c$, (3.10) becomes

$$\det(E + De^{-i\omega_c r} - i\omega_c I) = 0 \tag{3.12}$$

Solving this we have

$$\kappa_* = \frac{2\zeta^2\omega_c^2}{\omega_c^2 - 1} + \frac{\omega_c^2 - 1}{2} \qquad r = \frac{2}{\omega_c}\left\{\tan^{-1}\left(\frac{1-\omega_c^2}{2\zeta\omega_c}\right) + n\pi\right\} \tag{3.13}$$

For a fixed r, there exists many ω_c which satisfy the second equation of (3.13). And to each of those ω_c there exists corresponding κ_* which satisfies first equation of (3.13). Lowest of all such κ_* (call it κ_c)is where the first imaginary axis crossing occurs i.e. for $\kappa < \kappa_c$, all eigenvalues have negative real part.

Therefore, at $\kappa = \kappa_c$, one pair of eigenvalues, $\pm i\omega_c$ lie on imaginary axis, and all others have negative real part. Now, let P be the generalized eigenspace corresponding to the above eigenvalues. A basis Φ for P can be taken as

$$\Phi(\theta) = \begin{bmatrix} e^{i\omega_c\theta} & e^{-i\omega_c\theta} \\ i\omega_c e^{i\omega_c\theta} & -i\omega_c e^{-i\omega_c\theta} \end{bmatrix} \tag{3.14}$$

Let Ψ be the basis for generalized eigenspace asssociated with corresponding transposed equation, normalized by the condition $\langle\Psi, \Phi\rangle = I$, where the bilinear form $\langle\cdot,\cdot\rangle$ is given by

$$\begin{aligned}\langle\psi, \phi\rangle &\stackrel{\text{def}}{=} (\psi(0), \phi(0)) - \int_{\theta=-r}^{0}\int_{\tau=0}^{\theta}\psi(\tau-\theta)(E\delta(\theta) + D\delta(\theta + r))\phi(\tau)d\tau d\theta \\ &= (\psi(0), \phi(0)) - \int_{\tau=-r}^{0}\psi(\tau + r)D\phi(\tau)d\tau\end{aligned}$$

In the above equation $(\cdot,\cdot)$ stands for Hermite Inner Product. Ψ is found to be

$$\Psi(\tau) = \begin{bmatrix} \frac{2\zeta+i\omega_c}{N}e^{-i\omega_c\tau} & \frac{1}{N}e^{-i\omega_c\tau} \\ \frac{2\zeta-i\omega_c}{\overline{N}}e^{i\omega_c\tau} & \frac{1}{\overline{N}}e^{i\omega_c\tau} \end{bmatrix} \tag{3.15}$$

where $N = 2\zeta + 2i\omega_c + k_c r e^{-i\omega_c r}$ and $\overline{N}$ is complex conjugate of N. Now, $\mathcal{C}$ can be decomposed as $\mathcal{C} = P \oplus Q$ where

$$Q \stackrel{\text{def}}{=} \{\phi \in \mathcal{C} : \langle \Psi, \phi \rangle = 0\}$$

and any element $x_t \in \mathcal{C}$ can be written as $x_t = x_t^P + x_t^Q$, where $x_t \in P$ and $x_t^P = \Phi \langle \Psi, x_t \rangle$. And since, Φ is a basis for P, any x_t can be written as $x_t = \Phi z(t) + y_t$ where $z \in \mathbb{C}^2$ and y_t in Q. Also, since we have $AP \subset P$, there exists a constant matrix B defined by the relation $A\Phi = \Phi B$, which for the system under consideration, can be evaluated as

$$B = \begin{bmatrix} i\omega_c & 0 \\ 0 & -i\omega_c \end{bmatrix} \tag{3.16}$$

Now, for the purpose of the perturbed problem (3.3), as explained in chapter 1, we extend the state space involved. Let χ_0 be the matrix valued function with domain $[-r, 0]$

$$\chi_0 = \chi_0(\theta) = \begin{cases} 0_{2\times 2} & -r \le \theta < 0 \\ I_{2\times 2} & \theta = 0 \end{cases} \tag{3.17}$$

Now, let

$$\begin{aligned} \mathcal{BC} &\stackrel{\text{def}}{=} \mathcal{C} \oplus \operatorname{span}_{\mathbb{C}^2} \chi_0 \\ &= \left\{\phi : [-r, 0] \to \mathbb{C}^2, \phi \,\text{continuous on}[-r, 0)\text{with jump at } 0\right\} \end{aligned}$$

$$\mathcal{C}^1 = \left\{\phi : \phi \in \mathcal{C}, \frac{d\phi}{d\theta} \in \mathcal{C}\right\}$$

Any element in $\mathcal{BC}$ is of the form $\phi + X_0\alpha$ for some $\phi \in \mathcal{C}$. With the norm $||\phi + X_0\alpha||_{\mathcal{BC}} = ||\phi||_{\mathcal{C}} + ||\alpha||_{\mathbb{C}^2}$, $\mathcal{BC}$ is a Banach space. Now, define a new map $\hat{A} : \mathcal{C}^1 \to \mathcal{BC}$, defined by $\hat{A}\phi = \frac{d\phi}{d\theta} + \chi_0(L\phi - \frac{d\phi}{d\theta}|_{\theta=0})$. Then solution of

(3.3) satisfies

$$\frac{dx_t}{dt} = \hat{A}x_t + \chi_0 f(x_t) \tag{3.18}$$

The previous bilinear form can be extended to $\mathcal{C}' \times \mathcal{BC}$ by setting $(\Psi, X_0) = \Psi(0)$. It can be shown that A and $\hat{A}$ have the same spectrum. Define the projection operator $\hat{\pi} : \mathcal{BC} \to P$ as

$$\hat{\pi}(\phi + X_0\alpha) = \Phi[(\Psi, \phi) + \Psi(0)\alpha]$$

Then $\mathcal{BC} = P \oplus \operatorname{Ker}\hat{\pi}$. Now, for the hopf bifurcation scenario we are interested in, we can write $x_t = \Phi z(t) + y_t$ where $z(t) \in \mathbb{C}^2$ and $y_t \in \operatorname{Ker}\hat{\pi} \cap \mathcal{D}(\hat{A}) = Q \cap \mathcal{C}^1 \overset{\text{def}}{=} Q^1$. As $\hat{\pi}$ commutes with $\hat{A}$ in $\mathcal{C}^1$, (3.3) is equivalent to

$$\begin{aligned}\Phi\dot{z}(t) + \frac{dy_t}{dt} &= \Phi B z(t) + (I - \hat{\pi})\hat{\mathcal{A}}y_t \\ &+ \epsilon\, \Phi\Psi(0) F\left(\Phi z(t) + y_t, \epsilon\right) + \epsilon\,(I - \hat{\pi})\chi_0 F\left(\Phi z(t) + y_t, \xi(t), \epsilon\right).\end{aligned} \tag{3.19}$$

Now, projecting the above equation onto P and its complement in $\mathcal{BC}$ yields

$$\begin{aligned}\dot{z}(t) &= Bz(t) + \epsilon\,\Psi(0) F\left(\Phi z(t) + y_t, \xi(t), \epsilon\right) \\ \frac{dy_t}{dt} &= \hat{\mathcal{A}}y_t + \epsilon\,(I - \hat{\pi})\chi_0 F\left(\Phi z(t) + y_t, \xi(t), \epsilon\right),\end{aligned} \tag{3.20}$$

where we have once again used the fact that $\langle \Psi, y_t \rangle = 0$, Hence, the abstract ODE (3.18) in $\mathcal{BC}$ is equivalent to (3.20), and it is very important to realize that these *almost decoupled equations* are the *starting point* for the rest of our analysis. The second equation in (3.20) is interpreted as an equality for each $\theta \in [-r_0, 0]$. The spectrum of $(I - \pi)\hat{\mathcal{A}}$ is the same as $\sigma(A)$ excluding $\pm i\omega_c$.

3.3 Stochastic Stability of the trivial solution

Since our aim is to find asymptotic expansion of top lyapunov exponent of the RDS described by (3.20), we transform the variables into amplitude-phase variables given by

$$z_1(t) = e^{\rho(t)} e^{i\varphi(t)}, \qquad z_2(t) = e^{\rho(t)} e^{-i\varphi(t)} \qquad y_{1t} = e^{\rho(t)}\eta_{1t} \qquad y_{2t} = e^{\rho(t)}\eta_{2t} \tag{3.21}$$

Applying the transformation (3.21) to (3.20) yields the following set of equations for the logarithm of the amplitude, $\rho(t)$, the phase variable $\varphi(t)$ and η_t (with, recall, noise process $\xi(t)$),

$$\begin{aligned}
\dot\rho(t) &= q_\varepsilon(\xi(t), \varphi(t), \eta_t) = \varepsilon\, q(\xi(t), \varphi(t), \eta_t) \\
\dot\varphi(t) &= \omega_c + \varepsilon\, h^\varphi(\xi(t), \varphi(t), \eta_t) \\
\begin{pmatrix} \dot\eta_{1t} \\ \dot\eta_{2t} \end{pmatrix} &= \hat{\mathcal{A}}^Q \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix} + \varepsilon \chi_0^Q h^\eta(\xi(t), \varphi(t), \eta_t) - \varepsilon q(\xi(t), \varphi(t), \eta_t) \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \end{pmatrix}
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
q(\xi, \varphi, \eta) &= [K(\varphi) - M(\varphi, \eta)]\, r\, \sigma(\xi), \\
h^\varphi(\xi, \varphi, \eta) &= [H(\varphi) + \aleph(\varphi, \eta)]\, r\, \sigma(\xi), \\
h^\eta(\xi, \varphi, \eta) &= [L(\varphi) - \widetilde{P}(\eta)]\, r\, \sigma(\xi), \\
{\chi_0}^Q = \chi_0 - {\chi_0}^P,\ \ {\chi_0}^P(\theta) &\stackrel{\text{def}}{=} \hat\pi \chi_0(\theta) = \langle \Phi(\theta), \Psi(0) \rangle
\end{aligned}$$

Note that $q(\xi, \varphi, \eta)$ and $h^\varphi(\xi, \varphi, \eta)$ are *scalar-valued*, but $h^\eta(\xi, \varphi, \eta)$ is *vector-valued*. The φ dependance in the above expressions are given explicitly in terms of Fourier components as

$$\begin{aligned}
K(\varphi) &= K_0 + K_2\, e^{i\,2\varphi} + K_{-2}\, e^{-i\,2\varphi}, \\
M(\varphi, \eta) &= \left(M_1\, e^{i\,\varphi} + M_{-1}\, e^{-i\,\varphi} \right) P_2(\eta), \\
H(\varphi) &= H_0 - H_2\, e^{i\,2\varphi} - H_{-2}\, e^{-i\,2\varphi}, \\
\aleph(\varphi, \eta) &= \left(\aleph_1\, e^{i\,\varphi} + \aleph_{-1}\, e^{-i\,\varphi} \right) P_2(\eta_2), \\
L(\varphi) &= L_1\, e^{i\,\varphi} + L_{-1}\, e^{-i\,\varphi}
\end{aligned}$$

$$\begin{aligned}
K_0 &= \kappa\, \omega_c \ \mathrm{Im} \left(\frac{1}{N}\, e^{-i\omega_c r} \right) & K_2 &= \frac{\kappa\, \omega_c}{2i\overline{N}}\, e^{-i\omega_c r} & K_{-2} &= -\frac{\kappa\, \omega_c}{2iN}\, e^{i\omega_c r} \\
H_0 &= -\kappa\, \omega_c \ \mathrm{Re} \left(\frac{1}{N}\, e^{-i\omega_c r} \right) & H_2 &= -\frac{\kappa\, \omega_c}{2\overline{N}}\, e^{-i\omega_c r} & H_{-2} &= -\frac{\kappa\, \omega_c}{2N}\, e^{i\omega_c r}
\end{aligned}$$

$$
\begin{aligned}
M_1 &= \frac{\kappa}{2\overline{N}} & M_{-1} &= \frac{\kappa}{2N} \\
\aleph_1 &= \frac{\kappa}{2i\overline{N}} & \aleph_{-1} &= -\frac{\kappa}{2iN} \\
L_1 &= l_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} & L_{-1} &= l_{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
l_1 &= -\kappa i \omega_c e^{-i\omega_c r} & l_{-1} &= \kappa i \omega_c e^{i\omega_c r} \\
P_2(\eta) &= \eta_2(-r) & \widetilde{P}(\eta) &= \kappa \begin{pmatrix} 0 \\ \eta_2(-r) \end{pmatrix}
\end{aligned}
$$

Since $\xi(t)$ is a Markov process, $(\xi, \rho, \varphi, \eta)$ is Markov. In addition, the process (ξ, φ, η) does not depend on ρ, therefore it alone, without ρ, is Markovian, and its generator is given by

$$
\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon \mathcal{L}_1,
$$

where

$$
\begin{aligned}
\mathcal{L}_0 &= G + \omega_c \frac{\partial}{\partial \varphi} + \Big((\hat{\mathcal{A}}^Q \eta).\nabla \Big), \\
\mathcal{L}_1 &= h^\varphi(\xi, \varphi, \eta) \frac{\partial}{\partial \varphi} + \Big(\big({\chi_0}^Q \, h_1^\eta(\xi, \varphi, \eta) - \eta \, q_1(\xi, \varphi, \eta) \big) .\nabla \Big),
\end{aligned}
$$

Here, G, recall, is the generator of the noise process $\xi(t)$. Let $\mathbb{K} = \mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1)$. For any test function $f \in C^2(\mathbb{K})$, Fréchet differential of f in the direction $h \in (Q \cap \mathcal{C}^1)$ is given by the vector dot product $(\nabla f).h$, i.e. ∇f is the row-valued vector $(\frac{\partial f}{\partial \eta_1}, \frac{\partial f}{\partial \eta_2})$. Note that $\hat{\mathcal{A}}^Q \eta$ is vector valued. We indicate the first component by $(\hat{\mathcal{A}}^Q \eta)_1$ and similarly the second component by $(\hat{\mathcal{A}}^Q \eta)_2$. The $\Big((\hat{\mathcal{A}}^Q \eta).\nabla \Big)$ term in the above equation is interpreted as $(\hat{\mathcal{A}}^Q \eta)_1 \frac{\partial}{\partial \eta_1} + (\hat{\mathcal{A}}^Q \eta)_2 \frac{\partial}{\partial \eta_2}$.

Since $x_t = x_t^P + x_t^Q$, with x_t^Q in the eigenspace corresponding to eigenvalues with negative real part, it can be shown that the top Lyapunov exponent for the x_t is given by

$$
\lim_{t \to \infty} \frac{1}{t} \log \|x_t^P\|_P = \lim_{t \to \infty} \frac{1}{t} \log \|z(t)\|_{\mathbb{C}^2}
$$

The main aim now is to determine the top Lyapunov exponent based on the norm of the response

$$\|z(t;z_0)\| = \|z_0\| \exp\left\{\int_0^t q_\varepsilon(\xi(s),\varphi(s),\eta_s)ds\right\},$$

using the Furstenberg-Khasminskii formula given by

$$\lambda^\varepsilon = \lim_{t\to\infty}\frac{1}{t}\ln\|z(t;z_0)\| = \int_{\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1)} q_\varepsilon(\xi,\varphi,\eta)\,\mu_\varepsilon(d\xi,d\varphi,d\eta) \stackrel{\text{def}}{=} \langle q_\varepsilon,\mu_\varepsilon\rangle\,. \quad (3.23)$$

In what follows we will use the notation

$$\langle f,\mu\rangle = \int_{\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1)} f(\xi,\phi,\eta)\mu(d\xi,d\varphi,d\eta)$$

for the pairing of functions and measures on $\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1)$. When clear from the context, we will also freely identify $\langle f,\mu\rangle = \langle\mu,f\rangle$.

The direct procedure would be to solve $\mathcal{L}_\varepsilon^*\mu = 0$ for the joint invariant measure μ_ε, or equivalently,

$$\langle\mathcal{L}_\varepsilon f,\mu_\varepsilon\rangle = 0, \quad \forall f\in C^2\left(\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1)\right). \quad (3.24)$$

Since the exact joint invariant measure μ_ϵ is not easily found, we try to asymptotically approximate μ_ε:

$$\mu_\varepsilon(d\xi,d\varphi,d\eta) = \mu_0(d\xi,d\varphi,d\eta) + \varepsilon\mu_1(d\xi,d\varphi,d\eta) + \cdots \quad (3.25)$$

with suitable signed measures $\mu_0,\mu_1,\ldots$, and then to approximate the Lyapunov exponent: $\lambda^\varepsilon \approx \langle q_\varepsilon,\mu_0+\varepsilon\mu_1+\ldots\rangle$. With $q_\varepsilon = \varepsilon q$ we obtain

$$\lambda^\varepsilon = \varepsilon\langle q,\mu_0\rangle + \varepsilon^2\langle q,\mu_1\rangle + \varepsilon^3\langle q,\mu_2\rangle + \cdots \quad (3.26)$$

Since we are only interested in λ^ε up to second order terms, it suffices to find μ_0 and μ_1, provided that the remainder

$$r(\epsilon) \stackrel{\text{def}}{=} \lambda^\varepsilon - \left\{\varepsilon\langle q,\mu_0\rangle + \varepsilon^2\langle q,\mu_1\rangle\right\}$$

is of order $\mathscr{O}(\varepsilon^3)$

3.4 Asymptotic Approximation of The Maximal Lyapunov Exponent

We proceed in three steps. First we find μ_0 and μ_1 to approximate μ_ε (in some weak sense), second, we compute the terms $\langle q, \mu_0\rangle$, $\langle q, \mu_1\rangle$ and third, we calculate the order of the remainder term $r(\varepsilon)$. In this section, we discuss how to approach the three steps and find the answers in principle. The concrete calculations are done in section 3.5 under the assumption of a finite-state noise.

3.4.1 Expansion of the invariant measure μ_ε

Collecting the terms of the same ε-order and putting their sums to zero, we obtain from (3.24) and (3.25) together with $\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon\mathcal{L}_1$ the following equations for μ_0 and μ_1.

$$\begin{aligned} \langle \mathcal{L}_0 f,\, \mu_0\rangle \; &= 0 \\ \langle \mathcal{L}_0 f,\, \mu_1\rangle \; &= -\,\langle \mathcal{L}_1 f,\, \mu_0\rangle \end{aligned} \tag{3.27}$$

for all smooth functions on $\mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1)$.

The Solution to $\mathcal{O}(1)$

Consider the zeroth order equation in (3.27), that is, the weak form,

$$\langle \mathcal{L}_0\, f,\, \mu_0\rangle \; = 0 \quad \forall\, f \in C^2\left(\mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1)\right) \tag{3.28}$$

Based on the fact that all eigenvalues of $\hat{\mathcal{A}}^Q$ have negative real parts and the invariant measure $\nu(d\xi)$ satisfies $G^*\nu = 0$, we have

Lemma 3.4.1. *The product measure*

$$\mu_0(d\xi, d\varphi, d\eta) = \nu(d\xi)\, \frac{d\varphi}{2\pi}\, \delta_0(d\eta) \tag{3.29}$$

uniquely solves (3.28).

Proof. From (3.28), for all test functions $f \in C^2\left(\mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1)\right)$

$$
\begin{aligned}
\langle \mu_0,\, \mathcal{L}_0 f\rangle \;&=\; \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, \mathcal{L}_0 f\right\rangle \\
&= \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, \left(G(\xi)+\omega_c\frac{\partial}{\partial\varphi}\right) f\right\rangle \\
&\qquad + \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, \left((\hat{\mathcal{A}}^Q\eta).\nabla\right) f\right\rangle
\end{aligned}
$$

Taking apart each term in the first bracket yields the following three terms

$$
\begin{aligned}
&= \left\langle G^*(\xi)\nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, f\right\rangle + \int_{\mathbf{M}\times\mathbf{S}} \left\{\omega\frac{\partial f}{\partial\varphi}(\xi,\varphi,\eta)\right\}_{\eta=0} \nu(d\xi)\,\frac{d\varphi}{2\pi} \\
&\quad + \int_{\mathbf{M}\times\mathbf{S}} \left\{\left((\hat{\mathcal{A}}^Q\eta).\nabla\right) f(\xi,\varphi,\eta)\right\}_{\eta=0} \nu(d\xi)\,\frac{d\varphi}{2\pi}.
\end{aligned}
\tag{3.30}
$$

Since $\nu(d\xi)$ is the invariant measure and f is periodic in φ, each term in equation (3.30) is identically zero. Hence $\langle f, \mathcal{L}_0^*\mu_0\rangle = 0$ for all test functions f. The invariant measure μ_0 is unique, since by assumption, ν is the unique invariant measure of the noise process, the semigroup generated by $\omega_c\frac{\partial}{\partial\varphi}$ shifts the angle process all around $\mathbf{S}$, and $\hat{T}(t)\eta$ decays exponentially fast to zero for all $\eta \in V$. □

The Solution to $\mathcal{O}(\varepsilon)$

Now consider the first order equation in (3.27), that is,

$$
\langle \mu_1,\, \mathcal{L}_0 f\rangle \;=-\; \langle \mu_0,\, \mathcal{L}_1 f\rangle \quad \forall\, f \in C^2\left(\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1)\right) \tag{3.31}
$$

Lemma 3.4.2. *Let $\hat{r}(\xi,\varphi)$ and $\widetilde{r}(\xi,\varphi;\cdot)$ be solutions of*

$$
\begin{aligned}
\left(G^*(\xi) - \omega\frac{\partial}{\partial\varphi}\right)\left(\hat{r}(\xi,\varphi)\,\nu(d\xi)\right) \;&=\; \frac{\partial h_1^\varphi}{\partial\varphi}(\xi,\varphi,0)\nu(d\xi) \\
\left(G^*(\xi) - \omega\frac{\partial}{\partial\varphi} + \hat{\mathcal{A}}^Q\right)\left(\widetilde{r}(\xi,\varphi;\cdot)\,\nu(d\xi)\right) \;&=\; {\chi_0}^Q(\cdot)h_1^\eta(\xi,\varphi,0)\,\nu(d\xi),
\end{aligned}
\tag{3.32}
$$

respectively, then the measure

$$
\begin{aligned}
\mu_1(d\xi,d\varphi,d\eta) = \nu(d\xi)\,&\frac{d\varphi}{2\pi}\,\hat{r}(\xi,\varphi)\delta_0(d\eta) \\
&+ \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\frac{\partial\delta_0}{\partial\eta}\left(\widetilde{r}(\xi,\varphi;\cdot)\right)(d\eta)
\end{aligned}
\tag{3.33}
$$

solves (3.31).

Remark 3.4.3. This measure is defined with the help of Fréchet differentials. Fréchet differential of $f(\xi, \varphi, \eta)$ in the direction $h \in Q \cap \mathcal{C}^1$, is denoted by $f'(\xi, \varphi, \eta; h)$. Since the differential is linear in h, there exists an operator ∇ such that

$$f'(\xi, \varphi, \eta; h) = (\nabla f(\xi, \varphi, \eta)) .h$$

where the RHS is a vector dot product. Informally one can think of ∇f as the row vector $(\frac{\partial f}{\partial \eta_1}, \frac{\partial f}{\partial \eta_2})$. For any test function $f(\xi, \varphi, \eta)$

$$\left\langle f(\xi, \varphi, \eta), \frac{\partial \delta_0}{\partial \eta}\,(h)\,(d\eta)\right\rangle = -\,f'(\xi, \varphi, 0; h)$$

Note that $(h.\nabla) f(\xi, \varphi, \eta)$ would also mean $f'(\xi, \varphi, \eta; h)$

Proof. Taking first the right-hand-side of (3.31), for all test functions $f \in C^2\left(\mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1)\right)$, we have

$$\begin{aligned}
RHS = -\,\langle \mu_0,\, \mathcal{L}_1 f\rangle &= -\left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, h^{\varphi}(\xi, \varphi, \eta)\frac{\partial f}{\partial \varphi}\right\rangle \\
&\quad - \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\delta_0(d\eta),\, \left(\left({\chi_0}^Q(\theta)\, h^{\eta}(\xi, \varphi, \eta) - \eta\, q(\xi, \varphi, \eta)\right).\nabla\right)\, f\right\rangle \\
&= \int_{\mathbf{M}\times\mathbf{S}} \frac{\partial h^{\varphi}}{\partial \varphi}(\xi, \varphi, 0)\, f(\xi, \varphi, 0)\, \nu(d\xi)\,\frac{d\varphi}{2\pi} \\
&\quad - \int_{\mathbf{M}\times\mathbf{S}} \left\{\left(\left({\chi_0}^Q(\theta)\, h^{\eta}(\xi, \varphi, \eta)\right).\nabla\right) f(\xi, \varphi, \eta)\right\}_{\eta=0} \nu(d\xi)\,\frac{d\varphi}{2\pi}
\end{aligned}$$

where we have used the periodic boundary conditions of f in φ. Now taking the left-hand-side of (3.31) and substituting for μ_1 from (3.29) yields

$$\begin{aligned}
LHS = &\left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\hat{r}(\xi, \varphi)\delta_0(d\eta), \left(G(\xi) + \omega\frac{\partial}{\partial \varphi} + \left(\hat{\mathcal{A}}^Q \eta\right).\nabla\right) f\right\rangle \\
&+ \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\frac{\partial \delta_0}{\partial \eta}\,(\widetilde{r}(\xi, \varphi; \cdot))\,(d\eta), \left(G(\xi) + \omega\frac{\partial}{\partial \varphi} + \left(\hat{\mathcal{A}}^Q \eta\right).\nabla\right) f\right\rangle
\end{aligned} \tag{3.34}$$

We will consider each term in the above expression (3.34) separately. The first pairing in the LHS reduces to

$$
\begin{aligned}
&\left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\hat{r}(\xi,\varphi)\delta_0(d\eta), \left(G(\xi)+\omega\frac{\partial}{\partial\varphi}\right) f + \left(\left(\hat{\mathcal{A}}^Q\eta\right).\nabla\right) f \right\rangle \\
&= \left\langle \left(G^*(\xi)-\omega\frac{\partial}{\partial\varphi}\right) \left(\hat{r}(\xi,\varphi)\nu(d\xi)\right)\delta_0(d\eta), f(\xi,\varphi,\eta)\,\frac{d\varphi}{2\pi} \right\rangle \\
&+ \left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\hat{r}(\xi,\varphi)\delta_0(d\eta), \left(\left(\hat{\mathcal{A}}^Q\eta\right).\nabla\right) f \right\rangle \\
&= \int_{\mathbf{M}\times\mathbf{S}} \left\{ (G^*(\xi)-\omega\frac{\partial}{\partial\varphi}) \left(\hat{r}(\xi,\varphi)\,\nu(d\xi)\right)\; f(\xi,\varphi,\eta) \right\}_{\eta=0} \frac{d\varphi}{2\pi} \\
&+ \int_{\mathbf{M}\times\mathbf{S}} \left\{ \hat{r}(\xi,\varphi)\;\left(\left(\hat{\mathcal{A}}^Q\eta\right).\nabla\right) f(\xi,\varphi,\eta) \right\}_{\eta=0} \nu(d\xi)\,\frac{d\varphi}{2\pi}
\end{aligned}
$$

Now we will consider the second pairing in the LHS. We will use the fact that we can interchange the partial derivative operation in the mixed Fréchet derivatives of the Dirac measure δ_0 on $\mathcal{C}^1$ (with respect to different variables), that is,

$$
\left\langle \frac{d\varphi}{2\pi}\,\frac{\partial}{\partial\varphi}\left(\frac{\partial\delta_0}{\partial\eta}\left(\widetilde{r}(\xi,\varphi;\cdot)\right)(d\eta)\right), f \right\rangle = \left\langle \frac{d\varphi}{2\pi}\,\frac{\partial\delta_0}{\partial\eta}\left(\frac{\partial}{\partial\varphi}\left(\widetilde{r}(\xi,\varphi;\cdot)\right)\right)(d\eta), f \right\rangle
$$

Using the above fact, the second pairing in the LHS reduces to

$$
\begin{aligned}
&\left\langle \nu(d\xi)\,\frac{d\varphi}{2\pi}\,\frac{\partial\delta_0}{\partial\eta}\left(\widetilde{r}(\xi,\varphi;\cdot)\right)(d\eta), \left(G(\xi)+\omega\frac{\partial}{\partial\varphi}\right) f + \left(\left(\hat{\mathcal{A}}^Q\eta\right).\nabla\right) f \right\rangle \\
&= \left\langle \frac{\partial\delta_0}{\partial\eta}\left((G^*(\xi)-\omega\frac{\partial}{\partial\varphi})\;\left(\widetilde{r}(\xi,\varphi;\theta)\nu(d\xi)\right)\right)(d\eta), f(\xi,\varphi,\eta)\,\frac{d\varphi}{2\pi} \right\rangle \\
&- \left\langle \delta_0(d\eta), \left(\left(\widetilde{r}(\xi,\varphi;\theta)\right).\nabla\right)\left(\left(\left(\hat{\mathcal{A}}^Q\eta\right).\nabla\right) f(\xi,\varphi,\eta)\right)\nu(d\xi)\,\frac{d\varphi}{2\pi} \right\rangle \\
&= -\left\langle \delta_0(d\eta), \left(\left((G^*(\xi)-\omega\frac{\partial}{\partial\varphi})\;\left(\widetilde{r}(\xi,\varphi;\theta)\nu(d\xi)\right)\right).\nabla\right) f(\xi,\varphi,\eta)\,\frac{d\varphi}{2\pi} \right\rangle \\
&- \left\langle \delta_0(d\eta), \left(f''(\xi,\varphi,\eta;\hat{\mathcal{A}}^Q\eta,\widetilde{r}(\xi,\varphi;\theta)) + f'(\xi,\varphi,\eta;\hat{\mathcal{A}}^Q\widetilde{r}(\xi,\varphi;\theta))\right)\nu(d\xi)\,\frac{d\varphi}{2\pi} \right\rangle
\end{aligned}
$$

In the last term, we have denoted the second order Fréchet derivative with directions h and k as $f''(\xi,\varphi,\eta;h,k)$, which is bilinear with respect to (h,k) Since $h=\hat{\mathcal{A}}^Q\eta$ is linear in η, f'' vanishes when integrated against the Dirac measure at $\eta = 0$. Now taking the two LHS terms together yields

$$
\begin{aligned}
LHS = &-\int_{\mathbf{M}\times\mathbf{S}} \left\{ \left(\left((G^*(\xi) - \omega\frac{\partial}{\partial\varphi} + \hat{\mathcal{A}}^Q)\,(\widetilde{r}(\xi,\varphi;\theta)\nu(d\xi)) \right) .\nabla \right) f(\xi,\varphi,\eta) \right\}_{\eta=0} \frac{d\varphi}{2\pi} \\
&+ \int_{\mathbf{M}\times\mathbf{S}} \left\{ (G^*(\xi) - \omega\frac{\partial}{\partial\varphi})\,(\hat{r}(\xi,\varphi)\nu(d\xi))\ f(\xi,\varphi,\eta) \right\}_{\eta=0} \frac{d\varphi}{2\pi} \\
&+ \int_{\mathbf{M}\times\mathbf{S}} \left\{ \hat{r}(\xi,\varphi) \left(\left(\hat{\mathcal{A}}^Q\eta \right) .\nabla \right) f(\xi,\varphi,\eta) \right\}_{\eta=0} \nu(d\xi)\,\frac{d\varphi}{2\pi}
\end{aligned}
$$

By assumption of $\hat{r}$ and $\widetilde{r}$, the LHS equals RHS, and (3.33) is proven. □

We will solve the equations (3.32) in section 3.5 under the assumption of a finite-state Markov noise.

3.4.2 The remainder term $r(\varepsilon)$

Of course it is crucial to make sure that the expansion in (3.44) is, in fact, asymptotic. To this end, we will show that the remainder term

$$
\begin{aligned}
r(\varepsilon) &= \langle q_\varepsilon, \mu_\varepsilon \rangle - \langle q_\varepsilon, [\mu_0 + \varepsilon\mu_1] \rangle \\
&= \langle \varepsilon q, \mu_\varepsilon - [\mu_0 + \varepsilon\mu_1] \rangle
\end{aligned} \tag{3.35}
$$

is of order $\mathscr{O}(\varepsilon^3)$. This can be seen by modifying Theorem 3.1 in Arnold et al. [40] which leads to the following lemma.

Lemma 3.4.4. *Assume that for the given generator $\mathcal{L}_\varepsilon = \mathcal{L}_0 + \varepsilon\mathcal{L}_1$, the following holds true:*

I

$$\mathcal{L}_\varepsilon^*\mu_\varepsilon = 0, \quad \mathcal{L}_0^*\mu_0 = 0, \quad \mathcal{L}_0^*\mu_1 = -\mathcal{L}_1^*\mu_0$$

and the marginals of μ_ε, μ_0 and μ_1 on $\mathbf{M}$ are ν, ν and zero, respectively.

II There exists functions $F_0, F_1,$ and F_2 on $(\mathbf{M}\times\mathbf{S}\times(Q\cap\mathcal{C}^1))$ and functions $\widetilde{f}_0, \widetilde{f}_1,$ and $\widetilde{f}_2$ on $\mathbf{M}$ such that the sequence of Poisson-Type equations

$$
\begin{aligned}
\mathcal{L}_0F_0 &= -\widetilde{f}_0 \\
\mathcal{L}_0F_1 &= q - \widetilde{f}_1 - \mathcal{L}_1F_0 \\
\mathcal{L}_0F_2 &= -\widetilde{f}_2 - \mathcal{L}_1F_1
\end{aligned} \tag{3.36}
$$

are satisfied. Then

$$r(\varepsilon) = -\varepsilon^3 \langle \mathcal{L}_1 F_2,\, \mu_\varepsilon \rangle\,. \tag{3.37}$$

Assumption I are satisfied by the construction of the measures from (3.24), (3.28) and (3.31) in section 3.4. Moreover, due to the 2π-periodicity of the measures with respect to φ, the conditions on the marginals hold true. As to the assumption II, we need to solve the Poisson-Type equations (3.36) for F_0, F_1, and F_2. This can be done as in [40], which requires that that their right-hand sides belong to the range of $\mathcal{L}_0$. Since $q_0 = 0$, we may choose $\widetilde{f}_0 = 0$ and $F_0 = 0$. Then we have to consider only

$$\begin{aligned}
\mathcal{L}_0 F_1 &= q - \widetilde{f}_1 \\
\mathcal{L}_0 F_2 &= -\widetilde{f}_2 - \mathcal{L}_1 F_1,
\end{aligned}$$

and analogously to [40] we choose

$$\begin{aligned}
\widetilde{f}_1(\xi) &= \left\langle q, \frac{d\varphi}{2\pi}\delta_0(d\eta)\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} q(\xi,\varphi,0)d\varphi \\
&= K_0\sigma(\xi)
\end{aligned}$$

and

$$\widetilde{f}_2(\xi) = \left\langle -\mathcal{L}_1 F_1, \frac{d\varphi}{2\pi}\delta_0(d\eta)\right\rangle$$

(K_0 from (3.22) and following legend).

If we find F_1, F_2 such that $\mathcal{L}_1 F_2$ is bounded, then application of lemma 3.4.4 yields the required result, that is, the remainder term is bounded as

$$\begin{aligned}
|r(\varepsilon)| &= |\, \langle \varepsilon q\,,\, (\mu_\varepsilon - [\mu_0 + \varepsilon\mu_1]) \rangle \,| \\
&= \varepsilon^3 |\, \langle \mathcal{L}_1 F_2,\, \mu_\varepsilon \rangle \,| \le \varepsilon^3 C, \\
\text{with} \quad C &= \sup_{\xi,\varphi,\eta} |\mathcal{L}_1 F_2|.
\end{aligned} \tag{3.38}$$

3.5 Application to two state Markov Chain

Consider a stationary ergodic two-state Markov process $\xi(t)$ with state space $\mathbf{M} \stackrel{\text{def}}{=} \{1, 2\}$ and transition intensities $1 \xrightarrow{g_{12}} 2$, $2 \xrightarrow{g_{21}} 1$. The generator of the process is

$$G(\xi) = \begin{pmatrix} -g_{12} & g_{12} \\ g_{21} & -g_{21} \end{pmatrix}$$

The invariant measure of $\xi(t)$ is $\nu = \frac{1}{g}[g_{21}, g_{12}]^T$, where $g = g_{12} + g_{21}$.

Consider any test-function $f \in C^2 (\mathbf{M} \times \mathbf{S} \times (Q \cap \mathcal{C}^1))$. This can be considered as a vector of two functions $[f_1, f_2]^T$ where $f_1 = f(1, \varphi, \eta)$ and $f_2 = f(2, \varphi, \eta)$. Similarly measure of the process $\mu(\xi, \varphi, \eta)$ can also be considered as a vector having two components $[\mu^1(\varphi, \eta),\ \mu^2(\varphi, \eta)]^T$, and by $\langle \mu, f \rangle$ we mean the vector inner product $\mu^1 f_1 + \mu^2 f_2$.

Action of the generator $\mathcal{L}_\varepsilon$ of the (ξ, φ, η) process on a test-function f can be considered as follows: Rewrite the generator as

$$\begin{aligned}
\mathcal{L}_0 &= G + I\omega_c \frac{\partial}{\partial \varphi} + I\left((\hat{\mathcal{A}}^Q \eta).\nabla\right), \\
\mathcal{L}_1^1 &= h^\varphi(1, \varphi, \eta)\frac{\partial}{\partial \varphi} + \left(\left({\chi_0}^Q\, h^\eta(1, \varphi, \eta) - \eta\, q(1, \varphi, \eta)\right).\nabla\right), \\
\mathcal{L}_1^2 &= h^\varphi(2, \varphi, \eta)\frac{\partial}{\partial \varphi} + \left(\left({\chi_0}^Q\, h^\eta(2, \varphi, \eta) - \eta\, q(2, \varphi, \eta)\right).\nabla\right)
\end{aligned}$$

where I is 2×2 identity matrix. Note that $\left((\hat{\mathcal{A}}^Q \eta).\nabla\right) f(\xi, \varphi, \eta)$ is scalar-valued for each $\xi = 1, 2$. Therefore $\mathcal{L}_0 f$ can be treated as matrix multiplication

$$\begin{aligned}
\mathcal{L}_0 f(\xi, \varphi, \eta) = G \begin{pmatrix} f(1, \varphi, \eta) \\ f(2, \varphi, \eta) \end{pmatrix} &+ I\omega_c \frac{\partial}{\partial \varphi}\begin{pmatrix} f(1, \varphi, \eta) \\ f(2, \varphi, \eta) \end{pmatrix} \\
&+ I \begin{pmatrix} \left((\hat{\mathcal{A}}^Q \eta).\nabla\right) f(1, \varphi, \eta) \\ \left((\hat{\mathcal{A}}^Q \eta).\nabla\right) f(2, \varphi, \eta) \end{pmatrix}
\end{aligned} \tag{3.39}$$

and $\mathcal{L}_1 f(\xi, \varphi, \eta)$ is the column vector $[\mathcal{L}_1^1 f(1, \varphi, \eta),\ \mathcal{L}_1^2 f(2, \varphi, \eta)]^T$.

According to the lemma 3.4.1

$$\mu_0(\xi, \varphi, \eta) = \nu \frac{d\varphi}{2\pi}\delta_0(d\eta)$$

where ν is the invariant measure of $\xi(t)$ process. We write μ_0 as

$$\mu_0(\xi, \varphi, \eta) = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \frac{d\varphi}{2\pi}\delta_0(d\eta) \quad \text{with} \quad \nu_1 \stackrel{\text{def}}{=} \frac{g_{21}}{g} \quad \nu_2 \stackrel{\text{def}}{=} \frac{g_{12}}{g}$$

According to lemma 3.4.2

$$\mu_1(\xi,\varphi,\eta)=\begin{pmatrix}\nu_1\hat{r}_1(\varphi)\\ \nu_2\hat{r}_2(\varphi)\end{pmatrix}\frac{d\varphi}{2\pi}\delta_0(d\eta)+\begin{pmatrix}\nu_1\frac{\partial\delta_0}{\partial\eta}\left(\widetilde{r}_1(\varphi,\theta)\right)(d\eta)\\ \nu_2\frac{\partial\delta_0}{\partial\eta}\left(\widetilde{r}_2(\varphi,\theta)\right)(d\eta)\end{pmatrix}\frac{d\varphi}{2\pi}$$

where $\hat{r}_i(\varphi)$ satisfy the second of the Poisson equations (3.32)

$$\begin{aligned}(-g_{12}-\omega\frac{\partial}{\partial\varphi})\hat{r}_1+g_{12}\hat{r}_2 &= \frac{\partial h^{\varphi}}{\partial\varphi}(1,\varphi,0)\\ g_{21}\hat{r}_1+(-g_{21}-\omega\frac{\partial}{\partial\varphi})\hat{r}_2 &= \frac{\partial h^{\varphi}}{\partial\varphi}(2,\varphi,0)\end{aligned}\tag{3.40}$$

and $\widetilde{r}_i(\varphi,\theta)$ satisfy first of the Poisson equations (3.32)

$$\begin{aligned}\left(-g_{12}-\omega\frac{\partial}{\partial\varphi}+\hat{\mathcal{A}}^Q\right)\widetilde{r}_1+g_{12}\widetilde{r}_2 &= {\chi_0}^Q(\theta)h^{\eta}(1,\varphi,0)\\ g_{21}\widetilde{r}_1+\left(-g_{21}-\omega\frac{\partial}{\partial\varphi}+\hat{\mathcal{A}}^Q\right)\widetilde{r}_2 &= {\chi_0}^Q(\theta)h^{\eta}(2,\varphi,0)\end{aligned}\tag{3.41}$$

In the above equation, we have used the notation $\hat{r}_j(\varphi)\stackrel{\text{def}}{=}\hat{r}(j,\varphi,\theta)$, and $\widetilde{r}_j(\varphi,\theta)\stackrel{\text{def}}{=}\widetilde{r}(j,\varphi,\theta)$ for $j=1,2$. In arriving at the above equations we have used the fact that $\nu_1 g_{12}=\nu_2 g_{21}$.

Consider $\sigma(\xi)$. We write σ_1 for $\sigma(\xi=1)$ and σ_2 for $\sigma(\xi=2)$. Note that if $\sigma(\xi)$ is mean zero, then $\sigma_1\nu_1+\sigma_2\nu_2=\sigma_1 g_{21}+\sigma_2 g_{12}=0$.

Based on the form of the right hand side of (3.40),

$$\begin{aligned}\frac{\partial h^{\varphi}}{\partial\varphi}(\xi,\varphi,0) &= 2i\left(-H_2\,e^{2i\varphi}+H_{-2}\,e^{-2i\varphi}\right)r\sigma(\xi),\quad \xi=1,2\\ {\chi_0}^Q(\theta)h^{\eta}(\xi,\varphi,0) &= {\chi_0}^Q(\theta)\left(L_1\,e^{i\varphi}+L_{-1}\,e^{-i\varphi}\right)r\sigma(\xi),\quad \xi=1,2\end{aligned}\tag{3.42}$$

we assume the following form for the solution of $\hat{r}_i(\varphi)$ and $\widetilde{r}_i(\varphi,\theta)$.

$$\begin{pmatrix}\hat{r}_1(\varphi)\\ \hat{r}_2(\varphi)\end{pmatrix}=\begin{pmatrix}C_1\\ C_2\end{pmatrix}e^{2i\varphi}+\begin{pmatrix}C_{-1}\\ C_{-2}\end{pmatrix}e^{-2i\varphi}$$

$$
\begin{aligned}
\widetilde{r}_1(\varphi,\theta) &= \begin{pmatrix} B_1^{(1)}(\theta) \\ B_1^{(2)}(\theta) \end{pmatrix} e^{i\varphi} + \begin{pmatrix} B_{-1}^{(1)}(\theta) \\ B_{-1}^{(2)}(\theta) \end{pmatrix} e^{-i\varphi} \\
\widetilde{r}_2(\varphi,\theta) &= \begin{pmatrix} B_2^{(1)}(\theta) \\ B_2^{(2)}(\theta) \end{pmatrix} e^{i\varphi} + \begin{pmatrix} B_{-2}^{(1)}(\theta) \\ B_{-2}^{(2)}(\theta) \end{pmatrix} e^{-i\varphi}
\end{aligned}
$$

(number in the subscript is associated with the state ξ, sign in the subscript is indicative of sign of the exponential involved, and noting that $\widetilde{r}(\xi,\varphi,\eta)$ is vector valued for each ξ, the superscript indicates component). We also use the short notation

$$
\begin{aligned}
\widetilde{r}_1(\varphi,\theta) &= B_1(\theta)e^{i\varphi} + B_{-1}(\theta)e^{-i\varphi} \\
\widetilde{r}_2(\varphi,\theta) &= B_2(\theta)e^{i\varphi} + B_{-2}(\theta)e^{-i\varphi}
\end{aligned}
$$

where $B_1(\theta)$ is the column vector $\left(B_1^{(1)}(\theta),\, B_1^{(2)}(\theta)\right)^T$ and similarly others.

Equations (3.40) can be solved to yield

$$
C_1 = \frac{-4\omega H_2\sigma_1 r}{2i\omega(g_{12}+g_{21}) - 4\omega^2} \qquad C_2 = \frac{-4\omega H_2\sigma_2\, r}{2i\omega(g_{12}+g_{21}) - 4\omega^2}
$$

with C_{-i} being complex conjugates of C_i.

It will be shown in the subsequent steps that for the Lypunov exponent, only the difference $\widetilde{r}(1,\varphi,\eta) - \widetilde{r}(2,\varphi,\eta)$ matters and it can be given in terms of $B^+(\theta) \stackrel{\text{def}}{=} B_1(\theta) - B_2(\theta)$. From (3.41) it can be derived that $B^+(\theta)$ satisfies the following equation

$$
\frac{dB^+}{d\theta}(\theta) - (g_{12}+g_{21}+i\omega)B^+(\theta) = {\chi_0}^Q(\theta)L_1(\sigma_1-\sigma_2)\,r, \quad \theta \in [-1,0]
$$

Let

$$
Y = \kappa e^{-i\omega_c r} - \kappa e^{-(g+i\omega_c)r} + g(g+2\zeta+2i\omega_c)
$$

Then

$$
B_+(\theta) = -l_1\,(\sigma_1-\sigma_2)\,r\,\times
$$
$$
\left[\frac{1}{Y}\begin{pmatrix} 1 \\ g+i\omega_c \end{pmatrix} e^{(g+i\omega_c)\theta} - \frac{1}{N\,g}\begin{pmatrix} 1 \\ i\omega_c \end{pmatrix} e^{i\omega_c\theta} - \frac{1}{\overline{N}(g+2i\omega_c)}\begin{pmatrix} 1 \\ -i\omega_c \end{pmatrix} e^{-i\omega_c\theta}\right]
$$

Recall that

$$
\lambda^\varepsilon = \varepsilon\langle q,\mu_0\rangle + \varepsilon^2\langle q,\mu_1\rangle + \cdots \tag{3.43}
$$

and

$$q(\xi, \varphi, \eta) = [K(\varphi) - M(\varphi, \eta)]r\sigma(\xi)$$

Since $\mathbb{E}\sigma(\xi) = \int_M \sigma(\xi)\nu(d\xi) = 0$, by assumption, and μ_0 is a product measure, we obtain immediately

$$\langle q_1, \mu_0 \rangle = 0$$

So

$$\lambda^\varepsilon = \langle q_\varepsilon, \mu_\varepsilon \rangle = \varepsilon^2 \langle q, \mu_1 \rangle + \cdots \tag{3.44}$$

We now calculate the term $\langle q, \mu_1 \rangle$.

Consider $q(\xi, \varphi, \eta)$. We write $q_1 \stackrel{\text{def}}{=} q(1, \varphi, \eta)$ and $q_2 \stackrel{\text{def}}{=} q(2, \varphi, \eta)$. Now,

$$\begin{aligned} \langle q(\xi, \varphi, \eta), \mu_1 \rangle = & \left\langle \begin{pmatrix} \nu_1 \hat{r}_1 \\ \nu_2 \hat{r}_2 \end{pmatrix} \frac{d\varphi}{2\pi} \delta_0(d\eta), \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \\ & + \left\langle \begin{pmatrix} \nu_1 \frac{\partial \delta_0}{\partial \eta} (\widetilde{r}_1)(d\eta) \\ \nu_2 \frac{\partial \delta_0}{\partial \eta} (\widetilde{r}_2)(d\eta) \end{pmatrix} \frac{d\varphi}{2\pi}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right\rangle \end{aligned} \tag{3.45}$$

Note that $q_1(\varphi, \eta) = \sigma_1 r \left[K(\varphi) - M(\varphi, \eta)\right]$. Since in the first inner product, the integration is with respect to a Dirac measure $\delta_0(d\eta)$, the rest of the functions have to be evaluated at $\eta = 0$. Therefore $q_1(\varphi, 0) = \sigma_1 K(\varphi) r$. Similarly $q_2(\varphi, 0) = \sigma_2 K(\varphi) r$. Hence, the contribution from $\hat{r}$ terms in the (3.45), that is, first expression on the RHS, reduces to

$$\begin{aligned} \left\langle \begin{pmatrix} \nu_1 \hat{r}_1 \\ \nu_2 \hat{r}_2 \end{pmatrix} \frac{d\varphi}{2\pi} \delta_0(d\eta), \begin{pmatrix} \sigma_1 K(\varphi) r \\ \sigma_2 K(\varphi) r \end{pmatrix} \right\rangle = & \sigma_1 \nu_1 (K_{-2} C_1 + K_2 C_{-1}) r \\ & + \sigma_2 \nu_2 (K_{-2} C_2 + K_2 C_{-2}) r \end{aligned}$$

RHS of the above equation can be evaluated to be (denote it by $\hat{\lambda}$)

$$\hat{\lambda} = \frac{\sigma_1^2 g_{21} + \sigma_2^2 g_{12}}{(g_{12} + g_{21})^2 + 4\omega_c^2} \frac{\kappa^2 \omega_c^2 r^2}{N\overline{N}} \tag{3.46}$$

Now consider the second term in RHS of (3.45). Following the remark 3.4.3, this term reduces to

$$\left\langle \begin{pmatrix} \nu_1 \delta_0(d\eta) \frac{d\varphi}{2\pi} \\ \nu_2 \delta_0(d\eta) \frac{d\varphi}{2\pi} \end{pmatrix}, \begin{pmatrix} r\sigma_1 \frac{\partial M}{\partial \eta}(\varphi, \eta)(\widetilde{r}_1) \\ r\sigma_2 \frac{\partial M}{\partial \eta}(\varphi, \eta)(\widetilde{r}_2) \end{pmatrix} \right\rangle$$

$$\frac{\partial M}{\partial \eta}(\varphi,\eta)\,(\widetilde{r}_1) = \left(M_1\, e^{i\varphi} + M_{-1}\, e^{-i\varphi}\right) \frac{\partial P_2}{\partial \eta}(\eta)\,(\widetilde{r}_1)$$

Recalling that $P_2(\eta) = \eta_2(-r)$, we have

$$\begin{aligned}\frac{\partial M}{\partial \eta}(\eta)\,(\widetilde{r}_1) &= \left(M_1\, e^{i\varphi} + M_{-1}\, e^{-i\varphi}\right) \widetilde{r}_1^{(2)}(\varphi,-r)\\ &= \left(M_1\, e^{i\varphi} + M_{-1}\, e^{-i\varphi}\right) \left(B_1^{(2)}(-r)e^{i\varphi} + B_{-1}^{(2)}(-r)e^{-i\varphi}\right)\end{aligned}$$

Therefore, the second term in RHS of (3.45) reduces to (denote it by $\widetilde{\lambda}$)

$$\begin{aligned}\widetilde{\lambda} &= \sigma_1\nu_1 r\left[M_1 B_{-1}^{(2)}(-r) + M_{-1}B_1^{(2)}(-r)\right] + \sigma_2\nu_2 r\left[M_1 B_{-2}^{(2)}(-r) + M_{-1}B_2^{(2)}(-r)\right]\\ &= \sigma_1\nu_1 r\left[M_1\left(B_{-1}^{(2)}(-r) - B_{-2}^{(2)}(-r)\right) + M_{-1}\left(B_1^{(2)}(-r) - B_2^{(2)}(-r)\right)\right]\end{aligned}$$

The above equation shows that only the difference $\widetilde{r}(1,\varphi,\eta) - \widetilde{r}(2,\varphi,\eta)$ matters. Also, noting that the two expressions involved in the last part of the above equality are complex conjugates, it can be seen that it is enough to calculate $B_+(\theta) = B_1(\theta) - B_2(\theta)$ which we already did. Note that

$$M_{-1}l_1 = -\frac{\kappa^2 i\omega_c}{2N}e^{-i\omega_c r}$$

Let

$$Z = -\frac{r^2\kappa^2 i\omega_c}{2N}e^{-i\omega_c r}\left[-\frac{g+i\omega_c}{Y}e^{-(g+i\omega_c)r} + \frac{i\omega_c}{Ng}e^{-i\omega_c r} - \frac{i\omega_c}{\overline{N}(g+2i\omega_c)}e^{i\omega_c r}\right]$$

Then

$$\widetilde{\lambda} = \frac{\sigma_1^2 g_{21} + \sigma_2^2 g_{12}}{g_{12}+g_{21}}\, 2\,\mathrm{Re}\, Z$$

Recalling that $\langle q,\mu_1\rangle = \hat{\lambda} + \widetilde{\lambda}$, we get,

$$\langle q,\mu_1\rangle = \frac{\sigma_1^2 g_{21} + \sigma_2^2 g_{12}}{g_{12}+g_{21}}\ \mathrm{Re}\left[\frac{r^2\kappa^2 i\omega_c}{N}\left\{\frac{g+i\omega_c}{Y}e^{-gr} - \frac{i\omega_c}{Ng}\right\}e^{-2i\omega_c r}\right] \tag{3.47}$$

The effect of the third term in the expression for Z cancels the contribution of $\hat{\lambda}$. This nullifying is also found in the scalar SDDE case in [1]. Also top Lyapunov exponent has striking resemblance with that of the scalar SDDE case in [1].

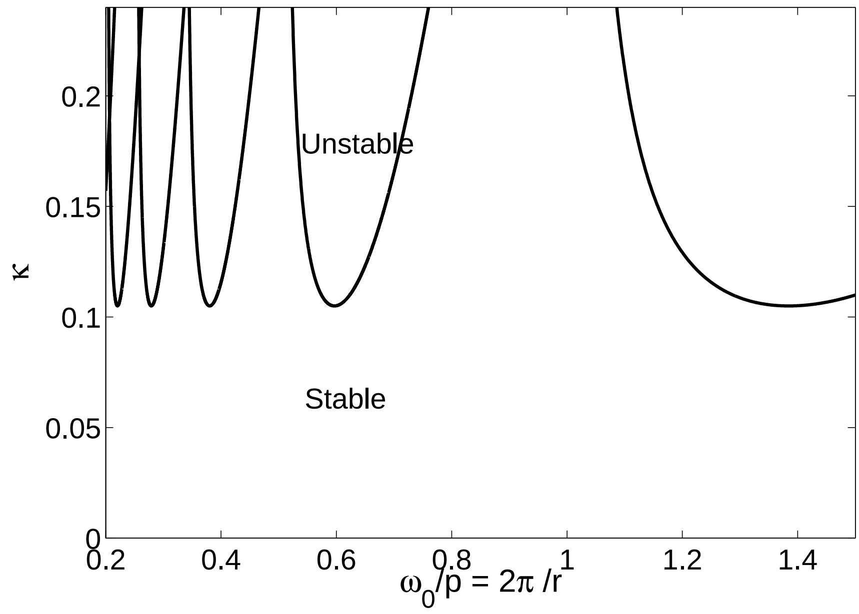

Figure 3.2: Satbility Chart for $\zeta = 0.05$

3.6 Application to Chatter Suppression

In this section, we attempt to answer whether chatter suppression is possible by modulating spindle speed according to two state markov noise. We compare our results with Yilmaz et al. [2] in which the MRSSV technique was proposed.

We take, as in [2], the damping coefficient $\zeta = 0.05$ and produce the stability chart in figure 3.2. The gap between $0.8 < \frac{\omega_0}{p} = \frac{2\pi}{r} < 1.0$ should be understood with caution. For $\kappa < 0.24$, as considered in the figure, tool motion is stable—but even in the above mentioned gap, for each r, there exists a κ beyond which the tool motion is unstable. However this $\kappa > 0.24$. Such a gap in subsequent figures should also be understood in this spirit.

In the figure 3.3 we show the effect of generator on the top Lyapunov exponent. We take

$$\epsilon\sigma_1 = -0.2 \qquad \epsilon\sigma_2 = 0.4 \qquad G = C \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

i.e. if nominal spindle speed is ω_0, then spindle speed remains $0.8\omega_0$ for two-thirds of the time and $1.4\omega_0$ for one-thirds of the time, switching between

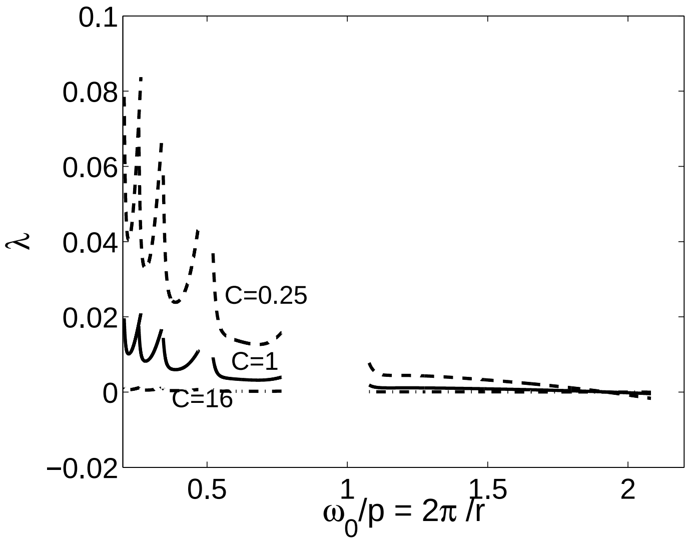

Figure 3.3: Effect of Generator, G, on the top Lyapunov Exponent λ

the two-states being modeled as a markov chain. The figure 3.3 shows the top Lyapunov exponent for different values of C. The figure shows that for C large, top Lyapunov exponent is nearly zero. Note that the mean holding time (time spent in one state before switching to other state) is inversely proportional to C. Hence, more rapid the switching, less is the influence of noise.

[2] reports that chatter suppression can be achieved by MRSSV technique. In this technique, the spindle speed is varied as piece-wise constant with jumps spaced uniformly in time. The amplitude of the signal is generated by a uniform distributed noise. In this chapter we considered the case where spindle speed is modulated as a two state markov chain. Though number of states available is less than in [2], though model of noise is different, we expect at least a qualitative agreement in the results. However, it can be seen from the figure 3.3 that the top Lyapunov exponent is positive and hence no stabilization could be achieved.

It is natural to wonder how much the stability curve is lowered due to the destabilization. Let κ_c be the critical value of κ for a fixed r. Assume stability boundary is lowered by a small amount i.e. $\kappa_c - \tilde{\kappa}$ be the new critical value when the spindle speed is varied according to two state markov chain, $\tilde{\kappa}$ being small. Let the top Lyapunov exponent obtained be λ. Let λ_0 be any

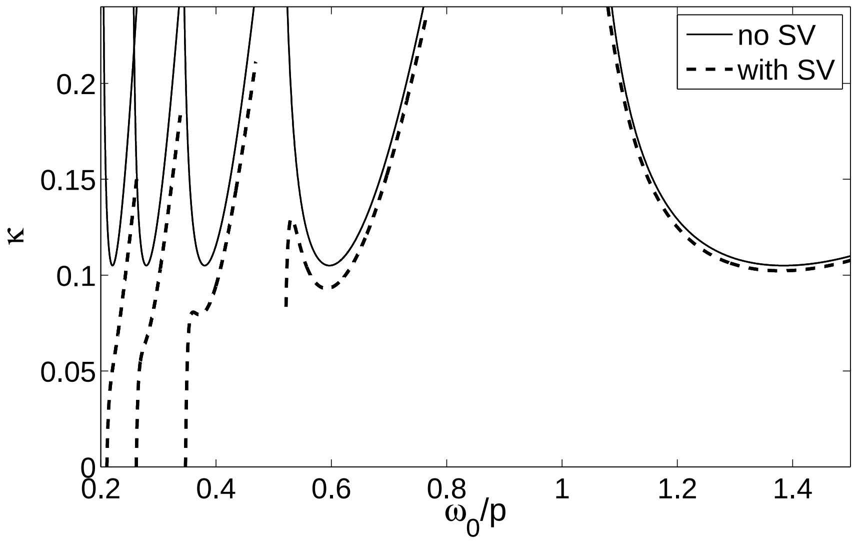

Figure 3.4: Lowering of stability boundary. G and σ same as in figure 3.3 with $C = 1$

eigenvalue of the system in absence of spindle speed modulation. Then

$$(\kappa_c - \tilde{\kappa})\delta_c = \lambda$$

where

$$\delta_c = \mathrm{Re}\left(\frac{d\lambda_0}{d\kappa}\right)\Bigg|_{\kappa=\kappa_c, \lambda_0 = i\omega_c}$$

We show the lowering of the stability boundary in the figure 3.4. The figure shows new boundary obtained as is from the above equation. Note that at some $\frac{\omega_0}{p}$ deviation from no noise case is very high. In such cases, the above equation should not be used because, the eigenvalues might no longer be close to $i\omega_c$. Note that this situation arises for $\frac{\omega_0}{p} < 0.6$ and it can be seen that this corresponds to large top Lyapunov exponent in the figure 3.3.

Consider the equation (3.2). We have

$$\dot{x}_2(t) = -(1+\kappa)\, x_1(t) + \kappa\, x_1(t-r) - 2\zeta\, x_2(t) - \epsilon\sigma(\xi)\, \kappa\, r\, x_2(t-r) \quad (3.48)$$

In order for the perturbation not to destroy the eigenvalue structure already present, we need $\epsilon\sigma\kappa r << 2\zeta$. Note that $\zeta = 0.05$ and $\kappa \geq 2\zeta(1+\zeta) = 0.105$. If spindle speed modulation is 10% then $\epsilon\sigma = 0.1$. For $\frac{\omega_0}{p} = 0.2$, we have $\epsilon\sigma\kappa r \approx 0.33$, whereas $2\zeta = 0.1$. Therefore, we conclude that for the range of $\frac{\omega_0}{p}$ considered, performing taylor expansion and retaining only the ϵ order terms in (3.2) leads to large inaccuracies.

3.7 Conclusion

Asymptotic expansion for top Lyapunov exponent is obtained. The top Lyapunov exponent being positive shows that varying spindle speed according to a two state Markov chain has a destabilizing effect. Nevertheless, the asymptotic results obtained are quantitatively inaccurate for explaining destabilization because the effect of perturbation by noise is not small in the regime of parameters considered.

CHAPTER 4

CONCLUSIONS

In chapter 2, we have studied the stability of gyroscopic systems with periodically varying delay. Our aim was to extend the work of Namachchivaya and Van Roessel [17] to explain the mechanism of chatter suppression in boring process. Variations in the delay were small and so we expanded the equation about a finite mean delay. Then we have augmented the system by considering the periodic fluctuation in delay as a state, thus converting a time-dependent delay system to state-dependent delay system. On the verge of instability, the system has a pair of eigenvalues on the imaginary axis and rest of them with negative real parts. Projecting the dynamics on to the eigenspace corresponding to the imaginary eigenvalues and employing center-manifold and normal form techniques, we explained the stabilizing mechanism and obtained analytical results for enlargement of stability boundary. Numerical simulations also show enlargement of stability boundaries but not to the extent given by analytical results.

In chapter 3, our aim was to extend the work of Namachchivaya and Wishtutz [1] to the case of a vector delay differential equation perturbed by a two state markov chain and also to check whether stabilization could be achieved by varying spindle speed as a two state Markov chain. Following [1] we illustrate procedure to find the asymptotic expansion for top Lyapunov exponent and calculate it explicitly to second order for the vector case. The top Lyapunov exponent obtained is positive and it shows that varying spindle speed according to a two state Markov chain has a destabilizing effect. Nevertheless, the asymptotic results obtained are quantitatively inaccurate for explaining destabilization because the effect of perturbation by noise is not small in the regime of parameters considered.

APPENDIX A

CALCULATIONS

A.1 Quadratic and Cubic nonlinearities

ξ_j^l is the entry in l^{th} row and j^{th} column of the matrix ξ. $\xi_j^{l\tau}$, γ_j^l are similarly defined.

$$\xi = \begin{bmatrix} 2\omega_r^2 & 0 & 2\zeta_1\omega_r & 2\omega_r \\ -2\zeta_2\omega_r\varpi & -2\omega_r & 2\omega_r^2 & 0 \end{bmatrix}$$

$$\xi^\tau = \begin{bmatrix} 0 & \mu_0\kappa\eta_{11}r & 0 & \mu_0\kappa\eta_{12}r \\ 0 & \mu_0\kappa\eta_{21}r & 0 & \mu_0\kappa\eta_{22}r \end{bmatrix}$$

$$\gamma = \begin{bmatrix} \omega_r^2 & 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 & 0 \end{bmatrix}$$

$$\begin{aligned} \gamma_j^{1\,\tau} &= \frac{r^2}{2}\left(D_{21}E_{2j} + D_{23}E_{4j}\right) - r\left(\delta_{j2}D_{21} + \delta_{j4}D_{23}\right) \\ \gamma_j^{2\,\tau} &= \frac{r^2}{2}\left(D_{41}E_{2j} + D_{43}E_{4j}\right) - r\left(\delta_{j2}D_{41} + \delta_{j4}D_{43}\right) \\ \gamma_j^{1\,2\tau} &= \frac{r^2}{2}\left(D_{21}D_{2j} + D_{23}D_{4j}\right) \\ \gamma_j^{2\,2\tau} &= \frac{r^2}{2}\left(D_{41}D_{2j} + D_{43}D_{4j}\right) \end{aligned}$$

$$\hat{f}_2^i = \sum_{|u|+|v|=2} \mathcal{Q}^i_{u_1u_2v_1v_2} x_1^{u_1}(t)x_3^{u_2}(t)x_1^{v_1}(t-r)x_3^{v_2}(t-r)$$

$$
\begin{aligned}
\mathcal{Q}^1_{u_1u_2v_1v_2} &= -\frac{1}{m\omega_1^2}\left[\mathbf{I}_{|u|=2}k^{(1)}_{u_1u_2} + \mathcal{G}\kappa\hat{\beta}_2\cos(\alpha+\beta)\right] \\
\mathcal{Q}^2_{u_1u_2v_1v_2} &= -\frac{1}{m\omega_1^2}\left[\mathbf{I}_{|u|=2}k^{(2)}_{u_1u_2} + \mathcal{G}\kappa\hat{\beta}_2\sin(\alpha+\beta)\right] \\
\mathcal{G} &= (\cos\alpha)^{u_1+v_1}(\sin\alpha)^{u_2+v_2}(-\mu)^{v_1+v_2}\binom{2}{|u|}\binom{|u|}{u_1}\binom{|v|}{v_1}
\end{aligned}
$$

where $\mathbf{I}$ stands for indicator function.

$$
\begin{aligned}
\hat{f}^i_3 = &\sum_{|u|+|v|=3}\tilde{\mathcal{Q}}^i_{u_1u_2v_1v_2}x_1^{u_1}(t)x_3^{u_2}(t)x_1^{v_1}(t-r)x_3^{v_2}(t-r) + \\
&+ \sum_{|u|+|v|=2}\mathcal{Q}^i_{u_1u_2v_1v_2}x_1^{u_1}(t)x_3^{u_2}(t)\times \\
&\times\left[v_1x_1^{v_1-1}(t-r)x_2(t-r)x_3^{v_2}(t-r) + v_2x_1^{v_1}(t-r)x_3^{v_2-1}(t-r)x_4(t-r)\right]x_6(t)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{Q}}^1_{u_1u_2v_1v_2} &= -\frac{1}{m\omega_1^2}\left[\mathbf{I}_{|u|=3}k^{(1)}_{u_1u_2} + \tilde{\mathcal{G}}\kappa\hat{\beta}_3\cos(\alpha+\beta)\right] \\
\tilde{\mathcal{Q}}^2_{u_1u_2v_1v_2} &= -\frac{1}{m\omega_1^2}\left[\mathbf{I}_{|u|=3}k^{(2)}_{u_1u_2} + \tilde{\mathcal{G}}\kappa\hat{\beta}_3\sin(\alpha+\beta)\right] \\
\tilde{\mathcal{G}} &= (\cos\alpha)^{u_1+v_1}(\sin\alpha)^{u_2+v_2}(-\mu)^{v_1+v_2}\binom{3}{|u|}\binom{|u|}{u_1}\binom{|v|}{v_1}
\end{aligned}
$$

A.2 Normalizing the Base $\Psi(\tau)$

A basis for the dual space was chosen to be $\Psi(\tau)$ as given in Eq.(2.23). But if the basis of dual space satisfies the normalizing condition $\langle\Psi,\Phi\rangle = I$, then further calculations would be simplified. To this cause; let the normalized basis be $\tilde{\Psi}(\tau)$. We take

$$
\tilde{\Psi} = \begin{pmatrix} a_{11}\Psi_1 + a_{12}\Psi_2 \\ a_{11}\Psi_1 + a_{12}\Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \tag{A.1}
$$

where Ψ_i is i^{th} row of Ψ given in Eq.(2.23) and the coefficients a_{ij} have to determined using the condition $\left\langle \tilde{\Psi}, \Phi \right\rangle = I$. This condition by Eq.(2.24) would read as

$$\tilde{\Psi}(0)\Phi(0) + \int_{-r}^{0} \tilde{\Psi}(\theta + r)D(\kappa_c)\Phi(\theta)d\theta = I \tag{A.2}$$

The coefficients a_{ij} can be determined by solving the above matrix equation. From now on we drop the tilde on $\tilde{\Psi}(\tau)$ and call it $\Psi(\tau)$.

A.3 Calculating the stability index

A.3.1 Normal Forms

We can expand the nonlinear terms in Eq.(2.36) (neglecting higher order terms) as follows:

$$\dot{z}(t) = \hat{B}z(t) + \Psi(0)F_2\left(\Phi z(t)\right) + \Psi(0)F_3\left(\Phi z(t)\right) + \Psi(0)\bar{D}_x F_2\left(\Phi z(t)\right) h\left(z(t);\theta\right) \tag{A.3}$$

Second and third terms in the RHS of the above equation correspond to quadratic and cubic nonlinearities and the last is due to corrections from center-manifold and $\left(\bar{D}_x F_2\right)_{ij}$ indicates $\frac{\partial F_2^i}{\partial x_j}$.

Say we have an equation of the following form:

$$\dot{z} = Bz + f_2(z) + f_3(z)$$

where f_2 and f_3 are respectively quadratic and cubic nonlinearities, B is a 4×4 diagonal matrix whose elements are λ_i. Each of f_2 and f_3 has 4 components and by $f_{2,s:m}$ we mean the term in s^{th} component of f_2 which is of the form $z^m = z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4}$. And by (λ, m) we mean $\sum \lambda_i m_i$. Suppose we perform the nonlinear transformation

$$z = u + g_2(u) + g_3(u)$$

Then u satisfies

$$\begin{aligned}\dot{u} = &\left[I - \bar{D}_u g_2 - \bar{D}_u g_3 + \bar{D}_u g_2 \bar{D}_u g_2 \ldots\right] \times \\ &\times \left[Bu + f_2 + Bg_2 + Bg_3 + f_3 + \bar{D}_u f_2 g_2 + \ldots\right]\end{aligned} \tag{A.4}$$

On rearranging terms we have

$$\begin{aligned}\dot{u} = Bu + \left[f_2 + Bg_2 - \bar{D}_u g_2 Bu\right] + \left[f_3 - \left(\bar{D}_u g_3 Bu - Bg_3\right)\right] + \\ + \bar{D}_u g_2 \left[\left(\bar{D}_u g_2 Bu - Bg_2\right) - f_2\right] + \bar{D}_u f_2 g_2\end{aligned}$$

If we choose the transformation g_2 such that $\bar{D}_u g_2 Bu - Bg_2 = f_2$ then the above equation reads

$$\dot{u} = Bu + \left[f_3 - \left(\bar{D}_u g_3 Bu - Bg_3\right)\right] + \bar{D}_u f_2 g_2$$

which can be achieved by choosing

$$g_{2,s:m} = \frac{f_{2,s:m}}{(\lambda, m) - \lambda_s}; \quad |m| = 2$$

Now note that the quadratic terms are eliminated from $\dot{u}$ equation. One might expect to eliminate the cubic terms also by choosing

$$g_{3,s:m} = \frac{f_{3,s:m} + \left(\bar{D}_u f_2 g_2\right)_{s:m}}{(\lambda, m) - \lambda_s}; \quad |m| = 3$$

Let

$$\mathcal{M}_s = \{m | (\lambda, m) - \lambda_s = 0; |m| = 3\}$$

$$\begin{aligned}\mathcal{M}_1 &= \{2100,\ 1011\} \\ \mathcal{M}_2 &= \{1200,\ 0111\} \\ \mathcal{M}_3 &= \{1110,\ 0021\} \\ \mathcal{M}_4 &= \{1101,\ 0012\}\end{aligned}$$

But for $m \in \mathcal{M}_s$ the denominator would be zero and hence terms of such form cannot eliminated. Of these 1011 and 0111, which are associated with $z_1 z_3 z_4$ and $z_2 z_3 z_4$, would result in linear terms on identifying that $z_3(t)$ is complex conjugate of $z_4(t)$ and that $z_3 z_4(t)$ is equal to square of the amplitude of the variation of spindle speed which is a constant. So, if we choose

$$g_{3,s:m} = \frac{f_{3,s:m} + \left(\bar{D}_u f_2 g_2\right)_{s:m}}{(\lambda, m) - \lambda_s}; \quad |m| = 3; \quad m \notin \mathcal{M}_s$$

$$g_{3,s:m} = 0; \quad |m| = 3; \quad m \in \mathcal{M}_s$$

then

$$\dot{u} = Bu + \tilde{g}_3$$

where $\tilde{g}_{3,s}$ contains terms of the form z^m such that $m \in \mathcal{M}_s$. These terms arise from $f_3 + \left(\bar{D}_u f_2 g_2\right)$. Of these terms, we are interested in $s : m = 1 : 1011$ because it leads to linear term and would contribute to further stabilization or destabilization. Also note that coefficient of $2 : 0111$ would be complex conjugate of $1 : 1011$.

Applying this discussion to Eq.(A.3), we would be interested in finding the coefficient of $1 : 1011$ term of

$$\begin{aligned} &\Psi(0) F_3 \left(\Phi z(t)\right) + \Psi(0) \bar{D}_x F_2 \left(\Phi z(t)\right) h \left(z(t); \theta\right) + \\ &+ \bar{D}_z \{\Psi(0) F_2 \left(\Phi z(t)\right)\} \frac{\{\Psi(0) F_2(\Phi z)\}_{s:m}}{(\lambda, m) - \lambda_s} \end{aligned} \tag{A.5}$$

Let's consider the third term in the above expression. The required coefficient is found to be $\mho_3$

$$\mho_3 = \sum_{v=1}^{2} \left({}^1\Upsilon_v^3 \; \Xi_v^4 + {}^1\Upsilon_v^4 \; \Xi_v^3\right)$$

where

$${}^u\Upsilon_v^n = \sum_{l=1}^{2} \Psi_{u\,2l}(0) \left[\sum_{j=1}^{4} (-1)^n (-i) \left(\xi_j^l \Phi_{jv}(0) + \xi_j^{l\tau} \Phi_{jv}(-r) \right) + (-1)^l \nu \omega_r \Phi_{5-2l\,v}(0) \right]$$

$$\Xi_v^n = \sum_{l=1}^{2} \Psi_{v\,2l}(0) \left[\sum_{j=1}^{4} (-1)^n (-i) \left(\xi_j^l \frac{\Phi_{j1}(0)}{\varrho} + \xi_j^{l\tau} \frac{\Phi_{j1}(-r)}{\varrho} \right) + (-1)^l \nu \omega_r \Phi_{5-2l\,1}(0) \right]$$

$$\varrho = (\lambda, m) - \lambda_v \qquad \lambda = (i\omega_c, -i\omega_c, i\nu, -i\nu) \qquad m = (1, 0, \delta_{3n}, \delta_{4n})$$

Now consider the first term in the expression (A.5). The required coefficient is found to be $\mho_1$

$$\mho_1 = \sum_{l=1}^{2}\sum_{j=1}^{4} 2\Psi_{1\,2l}(0)\left[\gamma_j^{l}\Phi_{j1}(0) + \gamma_j^{l\tau}\Phi_{j1}(-r) + \gamma_j^{l\,2\tau}\Phi_{j1}(-2r)\right]$$

Note that in the above equation we have used $\Phi(-2r)$ though θ was defined to be in the interval $[-r, 0]$. In that definition of θ our motivation was to illustrate the applicability of FDE theory without much complications. We could have as well defined $-2r \le \theta \le 0$ which would not change anything because there is no $t-2r$ delay in the linear terms of the equations of motion.

Now consider the second term in the expression (A.5) which involves center-manifold corrections. It can be deduced from the equations (2.39),(2.40) and (2.41) that the center-manifold terms of the form $w_{s:0011}(\theta)$,$w_{5:k}(\theta)$ and $w_{6:k}(\theta)$are zero. Using this result the required coefficient in the second term in the expression (A.5) is found to be $\mho_2 = \mho_2^3 + \mho_2^4$

$$\mho_2^3 = \sum_{l=1}^{2}\sum_{j=1}^{4} \Psi_{1\,2l}(0)\Big(\xi_j^{l}\Phi_{63}(0)y_{j:1001}(0) + \xi_j^{l\tau}\Phi_{63}(0)y_{j:1001}(-r)\Big) + \\ + \sum_{l=1}^{2}\Psi_{1\,2l}(0)(-1)^{l}\nu\omega_r\Phi_{53}(0)y_{5-2l:1001}(0)$$

$$\mho_2^4 = \sum_{l=1}^{2}\sum_{j=1}^{4} \Psi_{1\,2l}(0)\Big(\xi_j^{l}\Phi_{64}(0)y_{j:1010}(0) + \xi_j^{l\tau}\Phi_{64}(0)y_{j:1010}(-r)\Big) + \\ + \sum_{l=1}^{2}\Psi_{1\,2l}(0)(-1)^{l}\nu\omega_r\Phi_{54}(0)y_{5-2l:1010}(0)$$

A.3.2 Center-Manifold

From the expressions for $\mho_2$ and $\mho_3$, note that we need the center-manifold terms which are of the form $s : \delta_{p1}\delta_{p2}\delta_{q3}\delta_{q4}$, where p is either 1 or 2 and q is either 3 or 4. In all the expressions in this section let $s : pq$ denote $s : \delta_{p1}\delta_{p2}\delta_{q3}\delta_{q4}$. For example, $s : 24$ denotes $s : 0101$ and $s : 14$ denotes $s : 1001$. Let $\varsigma = [(-1)^{p}i\omega_c + (-1)^{q}i\nu]$. The equation Eq.(2.40) would read

$$w'_{s:pq}(\theta) + \varsigma w_{s:pq}(\theta) = \sum_{u=1}^{2} {}^{u}\Upsilon_p^{q}\,\Phi_{su}(\theta)$$

The general solution of which is

$$w_{s:pq}(\theta) = \frac{{}^1\Upsilon_p^q\ \Phi_{s1}(\theta)}{2i\omega_c(1-\delta_{p1}) + i\nu(-1)^q} + \frac{{}^2\Upsilon_p^q\ \Phi_{s2}(\theta)}{-2i\omega_c(1-\delta_{p2}) + i\nu(-1)^q} + C_{s:pq}e^{-\varsigma\theta}$$

where the third term on the RHS represents the solution of the homogenous equation and the first two terms represent the particular solution. Let the particular solution be denoted by $W_{s:pq}(\theta)$. The constants $C_{s:pq}$ should be obtained from the boundary conditions Eq.(2.41). Let

$${}^l\aleph_p^q = \sum_{j=1}^{4}(-1)^q(-i)\left(\xi_j^l\Phi_{jp}(0) + \xi_j^{l\tau}\Phi_{jp}(-r)\right) + (-1)^l\nu\omega_r\Phi_{5-2l\,p}(0)$$

The RHS of Eq.(2.41) is given by

$$V_{s:pq} = \delta_{s2}\,{}^1\aleph_p^q + \delta_{s4}\,{}^2\aleph_p^q - \sum_{u=1}^{2}{}^u\Upsilon_p^q\ \Phi_{su}(0)$$

Let V_{pq} be the 4×1 column matrix whose entries are $\{V_{s:pq}\}_{s=1:4}$. Let $W_{pq}(0)$ and $W_{pq}(-r)$ be the 4×1 column matrices whose entries are $\{W_{s:pq}(0)\}_{s=1:4}$ and $\{W_{s:pq}(-r)\}_{s=1:4}$respectively. Let C_{pq} be the 4×1 column matrix whose entries are $\{C_{s:pq}\}_{s=1:4}$. Let $\tilde{D}$ and $\tilde{E}$ denote the top left 4×4 portion of the matrices D and E respectively. Let I be the 4×4 identity matrix. Then the Eq.(2.41) can be written as:

$$(-\varsigma I - \tilde{E})\left[W_{pq}(0) + C_{pq}\right] - \tilde{D}\left[W_{pq}(-r) + e^{\varsigma r}C_{pq}\right] = V_{pq}$$

which can be solved as:

$$C_{pq} = \left(-\varsigma I - \tilde{E} - \tilde{D}e^{\varsigma r}\right)^{-1}\left[V_{pq} + \left(\tilde{E} + \varsigma I\right)W_{pq}(0) + \tilde{D}W_{pq}(-r)\right]$$

A.4 Crossing Condition

If λ is a simple eigenvalue of a matrix $A(\alpha)$ which depends on parameter α, then on applying Jacobi's formula for derivative of determinant of a matrix, we get

$$\frac{d\lambda}{d\alpha} = \frac{u^T \frac{dA}{d\alpha} v}{u^T v}$$

where u^T is row eigenvector and v is column eigenvector of matrix A corresponding to λ. Applying this to the matrix $E(\kappa) + D(\kappa)e^{-\lambda r}$ we get

$$\frac{d\lambda}{d\kappa} = \frac{u^T (\frac{dE}{d\kappa} + e^{-\lambda r} \frac{dD}{d\kappa}) v}{u^T (I + re^{-\lambda r} D) v}$$

evaluated at $\lambda = i\omega_c$, $\kappa = \kappa_c$. Here u^T is the first row of $\Psi(0)$ and v is the first column of $\Phi(0)$. The real part of $\frac{d\lambda}{d\kappa}$ is δ_c.

REFERENCES

[1] N. Sri Namachchivaya and V. Wihstutz, “Almost sure asymptotic stability of scalar stochastic delay equations: Finite state markov process,” in preparation.

[2] A. Yilmaz, E. Al-Regib, and J. Ni, “Machine tool chatter suppression by multi-level random spindle speed variation,” *Journal of Manufacturing Science and Engineering*, vol. 124, no. 2, pp. 208–216, 2002.

[3] M. Fofana, “Delay dynamical systems with applications to machine-tool chatter,” Ph.D. dissertation, University of Waterloo, 1993.

[4] A. H. Nayfeh, C. M. Chin, and J. Pratt, “Perturbation methods in nonlinear dynamics: application to machining dynamics,” *Journal of Manufacturing Science and Engineering, Transactions of the ASME*, vol. 119, pp. 485–493, 1997.

[5] Y. Liang, “Nonlinear dynamics in manufacturing process,” M.S. thesis, University of Illinois, 2000.

[6] T. Kalmár-Nagy, G. Stépán, and F. C. Moon, “Subcritical Hopf bifurcation in the delay equation model for machine tool vibrations,” *Nonlinear Dynamics*, vol. 26, pp. 121–142, 2001.

[7] H. M. Shi and S. A. Tobias, “Theory of finite amplitude machine tool instability,” *International Journal of Machine Tool Design and Research*, vol. 24, no. 1, pp. 45–69, 1984.

[8] F. C. Moon, *Dynamics and Chaos in Manufacturing Process.* New York: John Wiley & Sons Inc., 1998.

[9] T. Stoferle and H. Grab, “Vermeiden von ratterschwingungen durch periodische drehzahlanderung,” *Werkstatt und Betrieb*, vol. 105, pp. 727–730, 1972.

[10] S. C. Lin, S. G. Kapoor, and R. E. DeVor, “The effects of variable speed cutting on vibration control in face milling,” *Journal of Engineering for Industry, Transactions of the ASME*, vol. 112, pp. 1–11, 1990.

[11] T. C. Tsao, M. W. McCarthy, and S. G. Kapoor, "A new approach to stability analysis of variable speed machining systems," *International Journal of Machine Tools Manufacture*, vol. 33, pp. 791–808, 1993.

[12] S. Jayaram, S. G. Kapoor, and R. E. DeVor, "Analytical stability analysis of variable speed machining," *Journal of Manufacturing Science and Engineering, Transactions of the ASME*, vol. 122, pp. 391–397, 2000.

[13] T. Inamura and T. Sata, "Stability analysis of cutting under varying spindle speed," *Journal of the Faculty of Engineering, University of Tokyo-B*, vol. 23, pp. 13–29, 1975.

[14] T. Hosi and M. Sato, "Study of practical application of fluctuating speed cutting for regenerative chatter control," *Annals of the CIRP*, vol. 25, pp. 175–179, 1977.

[15] J. S. Sexton, R. D. Milne, and B. J. Stone, "The stability of machining with continuously varying spindle speed," *Applied Math. Modeling*, vol. 1, pp. 311–318, 1977.

[16] J. S. Sexton and B. J. Stone, "The stability of machining with continuously varying spindle speed," *Annals of the CIRP*, vol. 27, pp. 321–326, 1978.

[17] N. Sri Namachchivaya and H. J. Van Roessel, "A center-manifold analysis of variable speed machining," *Dynamical Systems*, vol. 18, no. 3, pp. 245–270, 2003.

[18] L. Vedula, "Dynamics and stability of parametrically excited gyroscopic systems," Ph.D. dissertation, University of Illinois, 2005.

[19] J. K. Hale and S. J. Verduyn-Lunel, *Introduction to Functional Differential Equations*. New York: Springer-Verlag, 1993.

[20] S. N. Chow and J. Mallet-Paret, "Integral averaging and bifurcation," *Journal of Differential Equations*, vol. 26, pp. 112–159, 1977.

[21] S. Tobias and W. Fishwick, "Theory of regenerative machine tool chatter," *Engineer, London*, vol. 205, p. 199, 1958.

[22] R. A. Hallam and R. S. Allsopp, "The design, development and testing of a prototype boring dynamometer," *International Journal of Machine Tool Design Research*, vol. 24, no. 1, pp. 241–266, 1962.

[23] J. Tlusty, "Berechnung des rahmens der werkzeugmaschinen," *Schwerindustrie der Tsxhecoslovakei*, vol. 1, p. 8, 1955.

[24] J. Tlusty and M. Polacek, "The stability of the machine tool against self-excited vibration in machining," *ASME Production Engineering Research Conference*, pp. 454–465, 1963.

[25] E. W. Parker, "Dynamic stability of a cantilever boring bar with machined flats under regenerative conditions," *Journal of Mechanical and Engineering Science*, vol. 12, no. 2, pp. 104–115, 1970.

[26] G. Zhang and S. G. Kapoor, "Dynamic modeling and analysis of the boring machining system," *Journal of Engineering for Industry, Transactions of the ASME*, vol. 109, pp. 219–226, 1987.

[27] J. R. Pratt, "Vibration control for chatter suppression with application to boring bars," Ph.D. dissertation, Virginia Polytechnic Institute and State University, 1997.

[28] C. H. Chen and K. W. Wang, "An integrated approach toward the dynamic analysis of high-speed spindles: Part 2: Dynamics under moving end load," *ASME Journal of vibration and Acoustics*, vol. 116, pp. 506–513, 1994.

[29] C. J. Li, "Tool-tip displacement measurement, process modeling, and chatter avoidance in agile precision line boring," Ph.D. dissertation, University of Michigan, 1999.

[30] L. E. El'sgol'ts and S. B. Norkin, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments.* Academic Press, 1973.

[31] J. K. Hale, "Nonlinear oscillations in equations with delays," in *Nonlinear Oscillations in Biology*, F. C. Hoppensteadt, Ed. American Mathematical Society, 1979, pp. 157–185.

[32] B. D. Hassard, N. D. Kazarinoff, and Y.-H. Wan, "Theory and applications of hopf bifurcation," *London Mathematical Society*, vol. Lecture Note Series 41, Cambridge University Press, 1983.

[33] G. Stepan, *Retarded Dynamical Systems: Stability and Characteristic functions.* Harlow, UK: Longman, 1989.

[34] T. Faria and L. T. Magalhães, "Normal forms for retarted functional differential equations and applications to Bogdanov-Takens singularity," *Journal of Differential Equations*, vol. 122, pp. 201–224, 1995.

[35] N. Chafee, "A bifurcation problem for a functional differential equation of finitely retarded type," *Journal of Mathematical Analysis and Applications*, vol. 35, pp. 312–348, 1971.

[36] J. Carr, *Applications of Center Manifold Theory*, ser. Applied Mathematical Sciences. New York: Springer-Verlag, 1981, vol. 35.

[37] J. Bélair and S. A. Campbell, "Stability and bifurcations of equilibria in a multiple-delayed differential equation," *SIAM Journal of Applied Mathematics*, vol. 54, pp. 1402–1424, 1994.

[38] S. A. Campbell and J. Bélair, "Analytical and symbolically-assisted investigation of hopf-bifurcations in delay-differential equations," *Canadian Applied Mathematics Quarterly*, vol. 3, no. 2, pp. 137–154, 1995.

[39] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer-Verlag, 1983.

[40] L. Arnold, G. Papanicolaou, and V. Wihstutz, "Asymptotic analysis of Lyapunov exponent and rotation number of the random oscillator and applications," *SIAM Journal of Applied Mathematics*, vol. 46, no. 3, pp. 427–450, 1986.